EIGENFUNCTIONS AND VERY SINGULAR SIMILARITY SOLUTIONS OF ODD-ORDER NONLINEAR DISPERSION PDES

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Abstract. Asymptotic properties of solutions of the nonlinear dispersion equations

$$u_t = (|u|^n u)_{xxx} \quad \text{and} \quad u_t = (|u|^n u)_{xxx} - |u|^{p-1}u \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+, \tag{0.1}$$

where $n > 0$ and $p > n + 1$ are fixed exponents, and their higher-odd-order analogies are studied. The global in time similarity solutions, which lead to "nonlinear eigenfunctions" of the rescaled ODEs, are constructed. The basic mathematical tools include a "homotopy-deformation" approach, where the limit $n \to 0^+$ in the first equation in (0.1) turns out to be fruitful by reducing the problems to the linear one at $n = 0$, i.e.,

$$v_t = v_{xxx},$$

for which Hermitian spectral theory was developed earlier in [4], as well as other nonlinear operator and numerical methods. The nonlinear limit $n \to +\infty$, with the limit PDE

$$(\operatorname{sign} v)_t = v_{xxx}, \quad \text{in terms of the variable } v = |u|^n u,$$

admitting three almost explicit nonlinear eigenfunctions, has also been described.

For the second equation in (0.1), *very singular similarity solutions* (VSSs) are constructed. A "nonlinear bifurcation" phenomenon at critical values $\{p = p_l(n)\}_{l \ge 0}$ of the absorption exponents is discussed. In fact, regardless their wide spread in applications and clear significance, such third-order nonlinear dispersion PDEs are quite poorly represented in general PDE theory, unlike their direct "neighbouring" second and fourth-order quasilinear parabolic equations (the so-called *porous medium equations*, PMEs)

$$u_t = (|u|^n u)_{xx} \text{ (the PME–2)} \quad \text{and} \quad u_t = -(|u|^n u)_{xxxx} \text{ (the PME–4)}.$$

The main goal of the present paper is to essentially fill such a gap.

1. Introduction: nonlinear dispersion PDEs

1.1. **NDEs: global smooth similarity patterns.** In the present paper, we study asymptotic properties of *nonlinear dispersion equations* (NDEs) of the following form:

$$u_t = (-1)^{k+1} D_x^{2k+1}(|u|^n u) - |u|^{p-1}u \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+, \quad k = 1, 2, \dots, \tag{1.1}$$

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where $n > 0$ and $p > n + 1$ are fixed exponents. This continues the study began in [4] for $n = 0$, i.e., for the *semilinear* dispersion equation. A continuous "homotopy" path $n \to 0^+$ connecting the present study with that in [4] turns out to be rather effective.

Let us clarify from the beginning a general role and the main purpose of the present study in the theory of higher-order NDEs such as (1.1):

(i) We will study here a countable family of *sufficiently smooth* continuous similarity solutions of VSS (very singular) type, which are defined for all $t > 0$.

(ii) In addition, (1.1), as an equation *with nonlinear dispersion mechanism*, contains and describes other key singularity phenomena such as a complicated formation of various *shock* and smoother *rarefaction* waves, which appear from discontinuous data, as well as a general *nonuniqueness* of "entropy solutions" after shocks. We do not touch these difficult, even mathematically obscure, phenomena and refer to [8, 13, 5] for further details.

Overall, it may be said that the smooth similarity behaviour and asymptotic patterns studied here occur in the NDE (1.1) for large times, when the shock wave influence has already been settled down via evolution, and becomes negligible.

Concerning local existence and uniqueness theory, including "smoothing results" for NDEs (i.e., for solutions without shocks and stronger singularities), see references and results in [8]. Since we are dealing with some special exact similarity solutions of (1.1), we do not use any advanced results of local and/or global regularity and any shock-entropy theory here, and always tackle continuous solutions.

Various applications of nonlinear dispersion equations are characterized in detail in [13], so we will minimize extra references to previous physical and more formal investigations of such PDE models, and concentrate on their quite unusual mathematical aspects of our interest. It is worth mentioning here a couple of remarkable examples of NDEs from integrable PDE theory. The first one is the third-order *Harry Dym equation*

$$u_t = u^3 u_{xxx}\,, \tag{1.2}$$

a most exotic integrable soliton equation; see [14, § 4.7] for survey and references therein. The second one is the fifth-order *Kawamoto equation* [16]

$$u_t = u^5 u_{xxxxx} + 5\, u^4 u_x u_{xxxx} + 10\, u^5 u_{xx} u_{xxx},$$

which has higher-degree algebraic terms. Shock waves for the pure fifth-order NDEs

$$u_t = u^5 u_{xxxxx} \quad \text{and} \quad u_t = (|u|^n u)_{xxxxx},$$

are studied in [9]. Let us discuss further necessary aspects of NDEs and their standing in general PDE theory.

1.2. **Compactons in NDEs: compactly supported travelling waves.** Consider the higher odd-order nonlinear dispersion equation with another divergent lower-order term:

$$u_t = (-1)^{k+1} D_x^{2k+1}(|u|^n u) + (|u|^n u)_x. \tag{1.3}$$

Equation (1.3) is a generalization of the third-order *Rosenau–Hyman* (RH) *equation*

$$u_t = (u^2)_{xxx} + (u^2)_x, \tag{1.4}$$

which models the effect of nonlinear dispersion in the pattern formation of liquid drops (see [21]). For $n = k = 1$, (1.3) is the RH equation in classes of nonnegative solutions.

It is well known that the RH equation (1.4) possesses explicit moving compactly supported soliton-type solutions known as *compactons*. Compactons have the same structure as travelling wave solutions, given by

$$u_{\rm c}(x,t) = f(z), \quad \text{where} \quad z = x - \lambda t.$$

So looking at compacton solutions for (1.3), on substitution we have that

$$-\lambda f' = (-1)^{k+1} D_z^{2k+1}(|f|^n f) + (|f|^n f)'.$$

After integrating once with zero constant (i.e., zero "flux"), we find that $f(z)$ satisfies

$$-\lambda f = (-1)^{k+1} D_z^{2k}(|f|^n f) + |f|^n f. \tag{1.5}$$

For $k = 1$, the ODE (1.5) can be solved explicitly (see the expression (1.9) below), while for $k \geq 2$, this is a difficult variational problem, which admits various countable families of compactly supported oscillatory solutions $f(z)$ of changing sign, [12].

Whilst we note that compacton solutions may be found for nonlinear dispersion equations, for non-conservative and non-divergent NDEs such as (1.1), TW solutions may be irrelevant for classes of bounded solutions of the Cauchy problem. Therefore, instead we look to find other more complicated similarity solutions.

1.3. **A relation to blow-up in reaction-diffusion theory.** Surprisingly, the NDE (1.3) is somehow related to the parabolic even-order equations of reaction-diffusion type:

$$u_t = (-1)^{k+1} D_x^{2k}(|u|^n u) + |u|^n u \quad \big(u_t = (u^{n+1})_{xx} + u^{n+1} \text{ for } k=1, \ u \geq 0\big). \tag{1.6}$$

Equation (1.6) admits blow-up self-similar solutions of the separate form

$$u(x,t) = (T-t)^{-\frac{1}{n}} f(x), \tag{1.7}$$

where T is the finite blow-up time. This self-similarity reduces the PDE to the ODE for the similarity profile f easily obtained by substituting (1.7) to (1.3):

$$(-1)^{k+1} D_x^{2k}(|f|^n f) + |f|^n f = \tfrac{1}{n}\, f. \tag{1.8}$$

For $k = 1$, equation (1.8) possesses the explicit compactly supported solution

$$f(x) = \begin{cases} \big[\frac{2(n+1)}{n(n+2)} \cos^2\big(\frac{nx}{2(n+1)}\big)\big]^{\frac{1}{n}} & \text{for } |x| < \frac{\pi(n+1)}{n}, \\ 0 \ \text{ for } |x| \geq \frac{\pi(n+1)}{n}. \end{cases} \tag{1.9}$$

Thus, (1.7), (1.9) yields the so-called *standing wave blow-up solution* (S-regime of blow-up) of (1.3), which have compact support for all times $t \in (0,T)$. This exact *Zmitrenko–Kurdyumov blow-up solution* has been known since the middle of 1970s; see details in [22, Ch. 4]. As we have mentioned, for $k \geq 2$, (1.8) cannot be solved explicitly, but the ODE is shown to admit a countable set of compactly supported solutions obtained by variational Lusternik–Schnirel'man and fibering methods, [12].

Thus, comparing the ODEs (1.8) and (1.5), we see that these coincide provided the compacton velocity λ is given by

$$\lambda = -\tfrac{1}{n}. \tag{1.10}$$

In other words, this means that some principles of compacton propagation for such NDEs (1.3) are directly related to blow-up formation mechanisms for reaction-diffusion equations (1.6). Moreover, in both equations, such asymptotic structures are expected to be structurally stable (for $k = 1$ in (1.6), this has been proved, [22, p. 260]). This is indeed surprising, since both classes of nonlinear PDEs seem to be responsible for entirely different physical phenomena (to say nothing of their different mathematical essence).

The lower order case of equation (1.6) for $k = 1$, which is just a standard reaction-diffusion PDE, is fairly well understood. However, the third-order nonlinear dispersion equation in (1.3), also for the minimal value $k = 1$, has not been studied extensively and some of its basic governing mathematical principles are still relatively unknown.

1.4. **Two main model NDEs and layout of the paper.** Thus, we will construct global similarity solutions of the following two NDEs. The first one is the pure NDE

$$u_t = (-1)^{k+1} D_x^{2k+1}(|u|^n u) \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+, \quad n > 0, \tag{1.11}$$

which is studied in Sections 2–4. Our goal therein is to show that (1.11) admits an infinite *countable* family of self-similar solutions governed by "nonlinear eigenfunctions" of a rescaled operator. Moreover, in Section 3.6, we show that this countable family as $n \to 0^+$ can be described by eigenfunctions from Hermitian spectral theory of the non-self adjoint operator [4, § 4]

$$\mathbf{B} = (-1)^{k+1} D_y^{2k+1} + \tfrac{1}{2k+1}\, y D_y + \tfrac{1}{2k+1}\, I. \tag{1.12}$$

The above operator occurs after similar scaling of the *linear dispersion equation* (LDE)

$$u_t = (-1)^{k+1} D_x^{2k+1} u \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+; \quad k \ge 1, \tag{1.13}$$

so we suggest to describe nonlinear eigenfunctions of the NDE (1.11) by using linear theory.

In Section 4, we perform the opposite "nonlinear" limit $n \to +\infty$ in (1.11). Namely, we observe that the natural change of the independent variable leads to the following PDE:

$$|u|^n u = v \quad \Longrightarrow \quad \big(|v|^{-\frac{n}{n+1}} v\big)_t = (-1)^{k+1} D_x^{2k+1} v. \tag{1.14}$$

The formal limit $n \to +\infty$ leads to the following *limit NDE*:

$$(\operatorname{sign} v)_t = (-1)^{k+1} D_x^{2k+1} v, \tag{1.15}$$

which admits analogous similarity patterns. It turns out that, at least for $k = 1$, where (1.15) takes a particularly simple form

$$(\operatorname{sign} v)_t = v_{xxx}, \tag{1.16}$$

first three occurring ODEs for (1.16) admit reducing to an algebraic system, and we develop a geometric-algebraic approach to constructing first nonlinear eigenfunctions.

In Section 7, we return to the study of VSSs of the NDE with absorption (1.1), where our main goal is to justify existence of the so-called p-bifurcation branches of VSS solutions, which appear at some critical exponents $p = p_l(n) > n+1$, $l = 0, 1, 2, \ldots$.

2. Similarity solutions of the NDE ("nonlinear eigenfunctions")

Here, we consider the pure NDE (1.11), which is connected to (1.3), but now we do not have the convection-like term $(|u|^n u)_x$, which is negligible in our asymptotics. This non-linear dispersion equation may be compared with the even-order model, which represents the general higher-order porous medium equation (the PME–$2m$)

$$u_t = (-1)^{m+1} \Delta^m (|u|^n u) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad m \geq 2. \tag{2.1}$$

The classic PME–2, for $m = 1$, appears in a number of physical applications, such as fluid and gas flows, heat transfer, or (nonlinear) diffusion. Other applications have been proposed in mathematical biology, lubrication and boundary layer theory, and various other fields. Papers exploring delicate asymptotics for the PME include [10], where extra references are available. For nonlinear eigenfunctions of (2.1) with $m = 2$, $N = 1$, see [7] and key references therein. Overall, the higher-order case $m \geq 2$ in (2.1) has been studied much less in the mathematical literature and represents a number of difficult open problems.

2.1. **Self-similar solutions: towards a "nonlinear eigenvalue problem".**

Our NDE (1.11) admits standard similarity solutions given by

$$u_{\rm gl}(x,t) = t^{-\alpha} f(y), \quad \text{where} \quad y = x/t^{\beta}, \tag{2.2}$$

for some unknown real parameters α and β. After substitution (2.2) into the NDE, we obtain the ODE for the rescaled profile f,

$$-\alpha t^{-\alpha-1} f - \beta t^{-\alpha-1} f' y = (-1)^{k+1} t^{-\alpha(n+1)-\beta(2k+1)} D_y^{2k+1}(|f|^n f). \tag{2.3}$$

By equating powers of t, the parameter β can be found in terms of α and is given by

$$\beta = \tfrac{1-\alpha n}{2k+1} > 0 \quad \text{for} \quad \alpha < \tfrac{1}{n}, \tag{2.4}$$

so that now $\alpha \in \mathbb{R}$ remains the only unknown *nonlinear eigenvalue.*

Thus, the ODE (2.3) takes the form

$$\mathbf{A}(f,\alpha) \equiv (-1)^{k+1} D_y^{2k+1}(|f|^n f) + \tfrac{1-\alpha n}{2k+1}\, f' y + \alpha f = 0 \quad \text{in} \quad \mathbb{R}. \tag{2.5}$$

In order to formulate a suitable *nonlinear eigenvalue problem* for the pairs $\{\alpha_l, f_l\}_{l \geq 0}$ for the ODE (2.5), one should specify the "boundary" conditions at $y = \pm\infty$, which is one of the main goals of this paper. It turns out that, loosely speaking, such conditions can be formulated as follows: for some special values of the real eigenvalues $\{\alpha_l = \alpha_l(n) > 0\}_{l \geq 0}$, the corresponding nonlinear eigenfunctions

$$\begin{cases} f_l(y) \text{ have:} \quad \text{(i) finite left-hand interface } (f_l(y) \equiv 0 \text{ for } y \ll -1), \\ \text{and (ii) "minimal oscillatory" behaviour as } y \to +\infty. \end{cases} \tag{2.6}$$

Both such conditions will get proper explanations later on. Of course, at least for sufficiently large $l \geq 2k+1$, the problem (2.5), (2.6) exhibits all typical features of self-similarity of the *second kind* (the *first kind* corresponds to standard dimensional analysis of PDEs), a notion introduced by Ya.B. Zel'dovich in 1956 [24]. Unlike previous and known examples, in general, the eigenvalues $\alpha_l(n)$ (and hence $\beta_l(n)$) cannot be found explicitly from any dimensional analysis. Thus, admitted (real) values of $\alpha = \alpha_l(n) > 0$ at this stage are still unknown and play a role of "nonlinear eigenvalues".

In particular, our goal is to show by a combination of analytic, formal, and numerical tools that, for any $n > 0$, the eigenvalue problem (2.5), (2.6)

$$\text{has a countable set of pairs } \{\alpha_l, f_l(y)\}_{l \geq 0}: \qquad \mathbf{A}(f_l, \alpha_l) = 0. \tag{2.7}$$

For the linear case with $n = 0$, the corresponding linear non-self-adjoint Hermitian spectral problem was developed in [4, § 4]; we present and use these results below.

2.2. **Towards blow-up patterns.** The present analysis admits a natural duality: since the NDE (1.11) is invariant under *reflections*,

$$\begin{cases} x \mapsto -x, \\ t \mapsto T-t, \end{cases} \tag{2.8}$$

the global patterns (2.2) lead to the corresponding reflected *blow-up* ones:

$$u_{\rm bl}(x,t) = (T-t)^{-\alpha} f(y), \quad y = -\tfrac{x}{(T-t)^\beta}, \quad \text{where} \quad \beta = \tfrac{1-\alpha n}{2k+1} > 0. \tag{2.9}$$

Therefore, the nonlinear eigenvalue problem (2.5), (2.6) is assumed to describe both countable families of global and blow-up patterns for the NDE (1.11). In what follows, for simplicity, we will mainly use a global treatment of such asymptotic patterns.

2.3. **A homotopy path $n \to 0^+$ to Hermitian spectral theory: a route to countability of nonlinear eigenfunctions.** For $n = 0$, α gives the eigenvalues $\{\lambda_l\}$ in linear Hermitian spectral theory, [4, § 4]. Indeed, for $n = 0$, (2.5) reads

$$(-1)^{k+1} D_y^{2k+1} f + \tfrac{1}{2k+1}\, f'y + \alpha f \equiv \mathbf{B} f + \big(\alpha - \tfrac{1}{2k+1}\big) f = 0. \tag{2.10}$$

Therefore, this gives the eigenvalue equation for the linear operator $\mathbf{B}$ in (1.12):

$$\mathbf{B}\psi = \lambda \psi, \quad \text{where} \quad \lambda = -\alpha + \tfrac{1}{2k+1}. \tag{2.11}$$

By Hermitian spectral theory [4], this defines a countable set of eigenvalues for $n = 0$:

$$\alpha_l(0) = \tfrac{1+l}{2k+1}, \quad l = 0, 1, 2, \ldots. \tag{2.12}$$

The corresponding eigenfunctions are then the normalized derivatives of the rescaled kernel $F(y)$ of the fundamental solution of the dispersion operator $D_t + (-1)^k D_x^{2k+1}$:

$$\psi_l(y) = \tfrac{(-1)^l}{\sqrt{l!}}\, D_y^l F(y), \quad l \geq 0. \tag{2.13}$$

Moreover, the "adjoint" (not in the standard L^2-sense) operator

$$\mathbf{B}^* = (-1)^{k+1} D_y^{2k+1} - \tfrac{1}{2k+1}\, y D_y \tag{2.14}$$

has the same spectrum and eigenfunctions $\{\psi_l^*\}$, which are *generalized Hermite polynomials* given by (see [4, § 5.2])

$$\psi_l^*(y) = \tfrac{1}{\sqrt{l!}}\Big[y^l + (-1)^{k+1} \sum_{j=1}^{\lfloor \frac{|l|}{2k+1} \rfloor} \tfrac{1}{j!} D_y^{(2k+1)j} y^l\Big]. \tag{2.15}$$

Finally, the operator pair $\{\mathbf{B}, \mathbf{B}^*\}$ has a bi-orthonormal sets of eigenfunctions.

As a key conclusion, we expect that there exists a uniform and pointwise convergence as $n \to 0^+$ of the nonlinear eigenfunctions to the linear ones for $\mathbf{B}$ defined by (2.13). We postpone explanations of main aspects of such a branching until Section 3.6, when we will have gained enough understanding of basic nonlinear eigenfunction theory involved. However, a rigorous proof of such a branching of eigenfunctions at $n = 0$ is a hard open problem. Nevertheless, this n-branching approach still remains a unique analytically convincing fact towards existence of an *infinite countable* discrete set of nonlinear eigenfunctions of the problem (2.5). In this case, branching theory as $n \to 0^+$ can be developed along the same lines as for the PME–4 (2.1) with $m = 2$ [7, § 6]. Such a construction then uses a specific Hermitian spectral theory created in [4]; see Section 3.6.

2.4. **Conservation laws: explicit values of first nonlinear eigenvalues.** As a first pleasant surprise, it turns out that some first eigenvalues for any $n > 0$ can be calculated explicitly by conservation laws for the NDE. Assuming that the solution $u(x,t)$ is integrable in x over $\mathbb{R}$ (this is a formal assumption that can be got rid of), we have that (1.11) is conservative in mass, and so

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} u(x,t)\,\mathrm{d}x = 0. \tag{2.16}$$

For similarity solutions (2.2), we have that

$$\int_{\mathbb{R}} u(x,t)\,\mathrm{d}x = t^{\beta-\alpha} \int_{\mathbb{R}} f(y)\,\mathrm{d}y.$$

This satisfies (2.16), provided that, in addition to (2.4),

$$\beta - \alpha = 0 \quad \Longrightarrow \quad \alpha_0(n) = \tfrac{1}{(2k+1)+n}, \tag{2.17}$$

for non-zero rescaled mass $\int f \ne 0$. So, on substitution into (2.5),

$$(-1)^{k+1} D_y^{2k+1}(|f|^n f) + \tfrac{1}{(2k+1)+n}\, f' y + \tfrac{1}{(2k+1)+n}\, f = 0.$$

Integrating once with the zero constant (a zero-flux condition), we end up with the ODE

$$(-1)^{k+1} D_y^{2k}(|f|^n f) + \tfrac{1}{(2k+1)+n}\, f y = 0. \tag{2.18}$$

Note that, for $n = 0$, we have exactly the linear ODE (see [4])

$$(-1)^{k+1} F^{(2k)} + \tfrac{1}{2k+1}\, F y = 0 \quad \text{for} \quad y \in \mathbb{R}. \tag{2.19}$$

For convenience, we use in (2.18) the natural substitution

$$Y = |f|^n f \quad \Longrightarrow \quad f = |Y|^{-\frac{n}{n+1}} Y, \tag{2.20}$$

in order to remove nonlinearities in the highest differential. Substitution yields

$$(-1)^{k+1} D_y^{2k} Y + \tfrac{1}{(2k+1)+n}\, y |Y|^{-\frac{n}{n+1}} Y = 0. \tag{2.21}$$

Similarly, we have conservation of the first moment, with

$$\int_{\mathbb{R}} x u(x,t)\,\mathrm{d}x = t^{2\beta-\alpha} \int_{\mathbb{R}} y f(y)\,\mathrm{d}y.$$

Hence, we have that

$$2\beta - \alpha = 0 \quad \Longrightarrow \quad \alpha_1(n) = \tfrac{2}{(2k+1)+2n}. \tag{2.22}$$

This then gives the ODE

$$(-1)^{k+1} D_y^{2k+1}(|f|^n f) + \tfrac{1}{(2k+1)+2n}\, f' y + \tfrac{2}{(2k+1)+2n}\, f = 0. \tag{2.23}$$

However, we cannot simply integrate this equation, as we could before, to reduce the order of the ODE. Instead, we multiply (2.23) by y, so that

$$(-1)^{k+1} D_y^{2k+1}(|f|^n f) y + \tfrac{1}{(2k+1)+2n}\, f' y^2 + \tfrac{2}{(2k+1)+2n}\, f y = 0,$$

and now it is possible to integrate by parts, to obtain

$$(-1)^{k+1} D_y^{2k}(|f|^n f) y + (-1)^k D_y^{2k-1}(|f|^n f) + \tfrac{1}{(2k+1)+2n}\, f y^2 = 0. \tag{2.24}$$

We also look at conservation of the second moment,

$$\int_{\mathbb{R}} x^2 u(x,t)\,\mathrm{d}x = t^{3\beta-\alpha} \int_{\mathbb{R}} y^2 f(y)\,\mathrm{d}y.$$

This gives $3\beta - \alpha = 0$, so that, by (2.4),

$$\alpha_2(n) = \tfrac{3}{(2k+1)+3n} \quad \text{and} \quad (-1)^{k+1} D_y^{2k+1}(|f|^n f) + \tfrac{1}{(2k+1)+3n}\, f' y + \tfrac{3}{(2k+1)+3n}\, f = 0. \tag{2.25}$$

Similarly, we multiply by y^2 and integrate to reduce the order, to obtain the ODE

$$\begin{aligned} &(-1)^{k+1} D_y^{2k}(|f|^n f) y^2 + 2(-1)^k D_y^{2k-1}(|f|^n f) y \\ &\quad + 2(-1)^{k+1} D_y^{2k-2}(|f|^n f) + \tfrac{1}{(2k+1)+3n}\, f y^3 = 0. \end{aligned} \tag{2.26}$$

These three conservation laws in particular are important, as we can explicitly find the first three (second-order) equations for the case $k = 1$ (corresponding to the first three nonlinear eigenvalues), but not for others. The case $k = 1$ is essential, since it is much easier to develop theory for the lower-order case, as well as it is easier to solve numerically (see Section 2.5).

In general, for arbitrary $k \geq 1$, for all $l < 2k+1$, where l is the eigenvalue index as before, we have our moments conservation given by

$$\int_{\mathbb{R}} x^l u(x,t)\,\mathrm{d}x = t^{(l+1)\beta-\alpha} \int_{\mathbb{R}} y^l f(y)\,\mathrm{d}y.$$

Therefore, our nonlinear eigenvalues are represented by

$$(l+1)\beta - \alpha = 0 \quad \Longrightarrow \quad \alpha_l(n) = \tfrac{l+1}{(2k+1)+(l+1)n} \quad \text{for any} \quad 0 \leq l < 2k+1. \tag{2.27}$$

The corresponding ODEs, which can be integrated once for all such eigenvalues, are

$$(-1)^{k+1} D_y^{2k+1}(|f|^n f) + \tfrac{1}{(2k+1)+(l+1)n}\, f'y + \tfrac{l+1}{(2k+1)+(l+1)n}\, f = 0. \tag{2.28}$$

All of them admit reducing the order by multiplying by y^l and integrating by parts l times. The function $Y = |f|^n f$ then solved the semilinear PDE

$$(-1)^{k+1} Y^{(2k+1)} + \tfrac{1}{(2k+1)+(l+1)n}\, y(|Y|^{-\frac{n}{n+1}} Y)' + \tfrac{l+1}{(2k+1)+(l+1)n}\, |Y|^{-\frac{n}{n+1}} Y = 0. \tag{2.29}$$

Thus, for such NDEs, one can always obtain explicitly $2k+1$ n-branches of nonlinear eigenvalues (2.27), though of course the solvability of the corresponding eigenvalue equations remains a difficult open problem, especially for large l. However, (2.27) shows one important feature of this nonlinear eigenvalue theory: the n-branches (2.27) are *global* in $n > 0$, i.e., exist for all $n > 0$. Existence of such a global continuation, with no turning points, is one of the most difficult question in general nonlinear operator theory; see e.g., [18, 23]. Recall that the eigenvalue problem (2.5) is by no means variational, neither contains any monotone or coercive operators.

Next, we must admit that, for any larger $l \ge 2k+1$, we cannot find $\alpha_l(n)$ explicitly using conservation laws, and then, searching for n-branches of these nonlinear eigenfunctions, we either should rely on the above local homotopy approach as $n \to 0^+$, or use advanced numerical methods.

2.5. **Numerical construction of nonlinear eigenfunctions.** We look to use a shooting method in order to find reliable profiles for the NDE (1.11). First, considering solutions satisfying conservation of mass, we have our first "nonlinear eigenvalue" (for $l = 0$), where $n = 0$ corresponds to the linear kernel, $\psi_0 = F(y)$, i.e., $\lambda = 0$ in (2.11). Here the rescaled equation is given by (2.21).

Consider first the simpler case $k = 1$, when we have the second-order equation:

$$Y'' = -\tfrac{1}{n+3}\, |Y|^{-\frac{n}{n+1}} Y y. \tag{2.30}$$

Then, it can be shown that the left-hand interface is not oscillatory (in fact, the same is true for $n = 0$, i.e., for the Airy function). Hence, for small solutions of $Y(y)$ as $y \to y_0^+$, with some interface point $y_0 < 0$, we can approximate it by

$$Y(y) = C_0 (y - y_0)_+^{\tilde\alpha}\big(1 + o(1)\big), \tag{2.31}$$

for some constant $C_0 > 0$ and exponent $\tilde\alpha > 0$. Here we have used the standard notation

$$(\cdot)_+ = \max\{0, \cdot\}.$$

Substituting (2.31) into the ODE (2.30), we obtain the leading equality

$$\tilde\alpha(\tilde\alpha - 1) C_0 (y - y_0)^{\tilde\alpha - 2} = -\tfrac{1}{n+3}\, C_0^{\frac{1}{n+1}} (y - y_0)^{\frac{\tilde\alpha}{n+1}} y_0, \tag{2.32}$$

where we use $(y - y_0)$ to mean $(y - y_0)_+\big(1 + o(1)\big)$, as defined before. Hence, we must have from (2.32) that

$$\tilde\alpha = \tfrac{2(n+1)}{n},$$

with $2 < \tilde{\alpha} \le 4$ for $n \ge 1$. From this, the constant C_0 is given by

$$C_0 = \big(\tfrac{n^2|y_0|}{2(n+1)(n+2)(n+3)}\big)^{\frac{n+1}{n}} \quad \text{for any interface} \quad y_0 < 0.$$

Our solution then has the expansion

$$Y(y) = \big(\tfrac{n^2|y_0|}{2(n+1)(n+2)(n+3)}\big)^{\frac{n+1}{n}} (y-y_0)^{\frac{2(n+1)}{n}} (1+o(1)) \quad \text{as} \quad y \to y_0^+, \tag{2.33}$$

with the derivative expansion obtained similarly,

$$Y'(y) = \tfrac{2(n+1)}{n}\big(\tfrac{n^2|y_0|}{2(n+1)(n+2)(n+3)}\big)^{\frac{n+1}{n}} (y-y_0)^{\frac{n+2}{n}} (1+o(1)). \tag{2.34}$$

We use the `MatLab` IVP solver `ode15s` to plot our profiles. Taking an arbitrary initial (interface) point $y = y_0 < 0$, our solution and first derivative will be zero here. Since both initial values are zero at $y = y_0$, we will often find the solution $Y = 0$, whilst trying to solve numerically. In order to overcome this problem, we must look at some point $y_0 + \delta$ (for small $\delta \sim 10^{-3}$), close to this point, and after finding the derivative there, this is used as the initial condition.

Obviously due to the nature of the expansion (2.34), we must take a relatively large initial point (in our case we take $y_0 = -10$), in order for us to have an initial condition that is not negligible. Hence, the solution is a large rescaling of the "nonlinear fundamental solution" with the unit mass. Below are a few profiles that have been found, in which we have taken $\delta = 10^{-3}$. For $n \sim 0.5$, the derivative Y' is very small and larger negative interface points must be used to find reliable profiles. This makes comparison, between different values of n, more difficult.

In Figures 1–4, we show the first nonlinear eigenfunction $Y_0(y)$, with $k = 1$, for various values of $n = 3, 2, 1$, and 0.7. Note that this is precisely the profile that, as $n \to 0^+$, must converge to the rescaled kernel $F(y) \equiv \mathrm{Ai}\,(y)$, representing Airy's classic function satisfying (cf. (2.30) for $n = 0$)

$$F'' + \tfrac{1}{3}\, F y = 0, \quad \textstyle\int F = 1, \text{ where } F \in L^2_\rho(\mathbb{R}), \quad \rho(y) = \begin{cases} \mathrm{e}^{a|y|^{3/2}} & \text{for } y < 0, \\ \mathrm{e}^{-ay^{3/2}} & \text{for } y > 0, \end{cases} \tag{2.35}$$

where $a > 0$ is a small enough constant.

In general, for all values of l, we have, for the lower-order case of $k = 1$, that the same expansion (2.33) and (2.34) hold near finite interfaces. Here $l < 2k + 1$ and hence, for $k = 1$, we can only take $l = 0, 1, 2$ for explicitly given nonlinear eigenvalues.

For $l = 1$, we have from (2.24), with $Y(y) = |f|^n f$ and $k = 1$, that the eigenfunction $Y_1(y)$ solves the ODE

$$Y'' = \tfrac{1}{y}\, Y - \tfrac{1}{2n+3}\, |Y|^{-\frac{n}{n+1}} Y y. \tag{2.36}$$

For $l = 2$ and $k = 1$, we have from (2.26) that $Y_2(y)$ solves

$$Y'' = \tfrac{2}{y}\, Y' - \tfrac{2}{y^2}\, Y - \tfrac{1}{3n+3}\, |Y|^{-\frac{n}{n+1}} Y y. \tag{2.37}$$

We use these to plot the profiles of our NDE with $l = 1$ and $l = 2$, which, for $n = 0$, coincide with the derivatives $F'(y)$ and $F''(y)$ respectively, of the linear kernel $F(y)$.

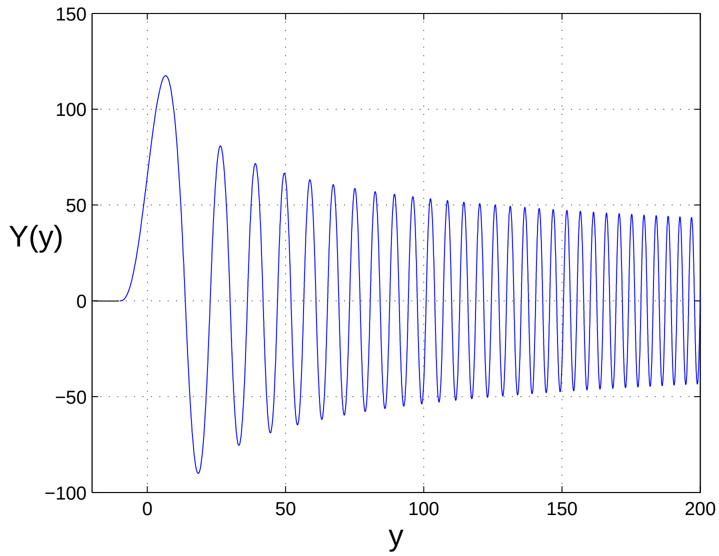

FIGURE 1. Rescaled solution $Y(y)$ of the ODE (2.30) for $k = 1$, $l = 0$, with $n = 3$.

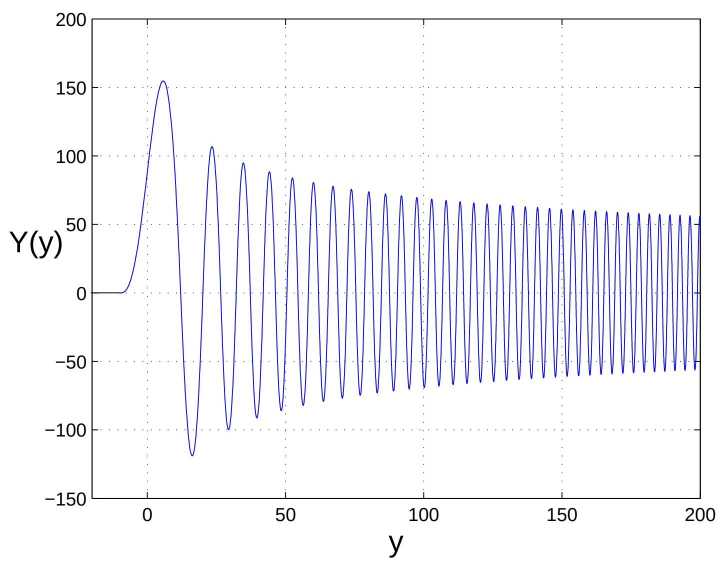

FIGURE 2. Rescaled solution $Y(y)$ of the ODE (2.30) for $k = 1$, $l = 0$, with $n = 2$.

Figures 5 and 6 show the "dipole-like" profiles for $l = 1$ as solutions of (2.36), while Figure 7 represents even more oscillatory third eigenfunction $f_2(y)$ for $l = 2$. Recall that, by a homotopy path as $n \to 0^+$, this function is expected to converge to the highly

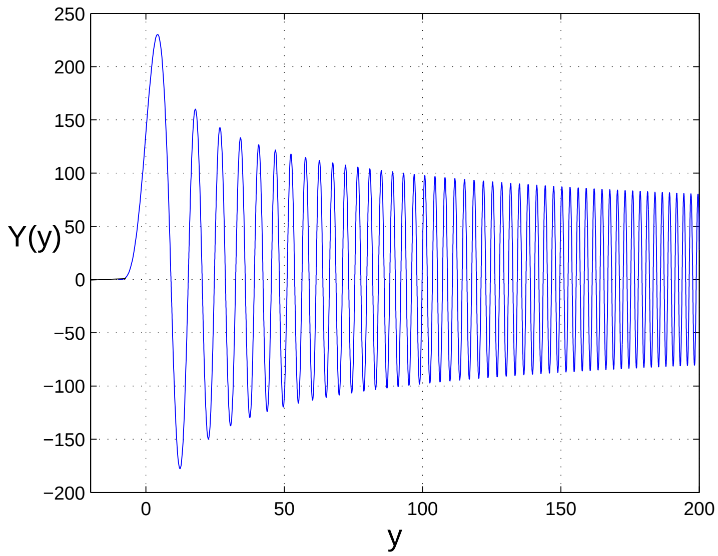

FIGURE 3. Rescaled solution $Y(y)$ of the ODE (2.30) for $k = 1$, $l = 0$, with $n = 1$.

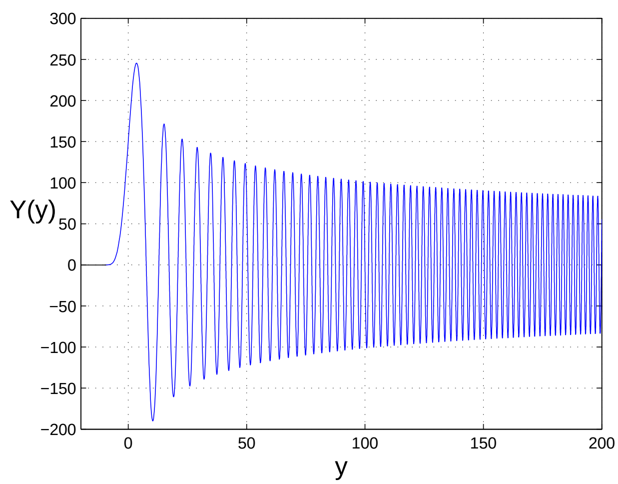

FIGURE 4. Rescaled solution $Y(y)$ of the ODE (2.30) for $k = 1$, $l = 0$, with $n = 0.7$.

oscillatory linear eigenfunction of the operator $\mathbf{B}$ in (1.12)

$$\psi_2(y) = \tfrac{1}{\sqrt{2}}\, F''(y).$$

For $k > 1$, the solutions $Y(y)$ are oscillatory (changing sign) as $y \to y_0^+$, so that expansions such as (2.31) must include *oscillatory components* as an extra multiplier:

$$Y(y) = (y - y_0)^{\hat{\alpha}}(\phi(s) + o(1)), \quad \text{where} \quad s = \ln(y - y_0), \tag{2.38}$$

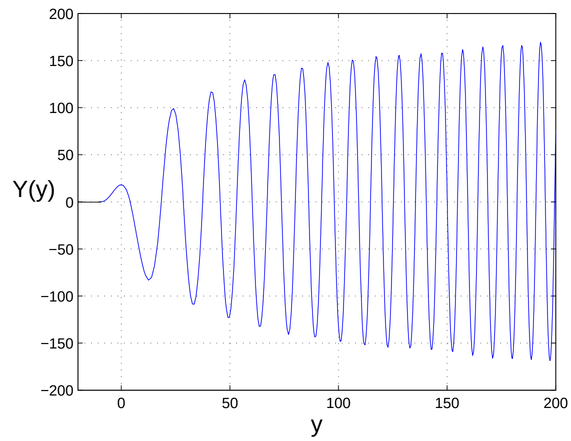

FIGURE 5. Rescaled solution $Y(y)$ of the ODE (2.36) for $k = 1$, $l = 1$, with $n = 3$.

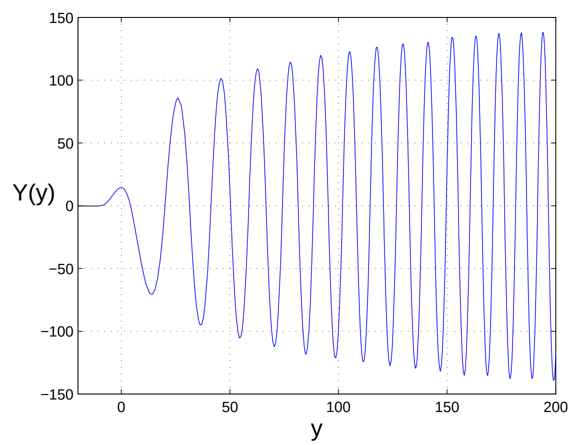

FIGURE 6. Rescaled solution $Y(y)$ of the ODE (2.36) for $k = 1$, $l = 1$, with $n = 4$.

and $\phi(s)$ is a periodic solution of a $(2k+1)$th-order ODE. Examples of such oscillatory structures (2.38) for $k = 2$ are presented in [14, pp. 186-192] for $k = 2$ and $k = 3$.

We will use a similar asymptotic approach in Section 3.5 in the opposite limit $y \to +\infty$.

3. On some mathematical aspects of similarity profiles

3.1. **Local existence and uniqueness for $k = 1$.** We need to show that the above numerical construction can be justified, by carefully verifying some rigorous aspects of the asymptotic analysis. We first apply a fixed point approach for the equivalent nonlinear integral equation to prove the expansion (2.33) near the finite left-hand interface.

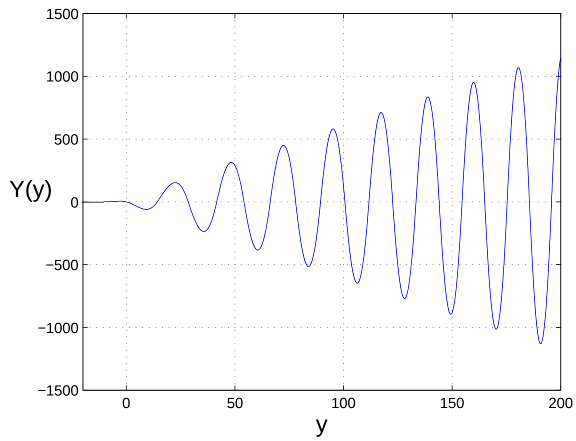

FIGURE 7. Rescaled solution $Y(y)$ for $k=1$, $l=2$ of the ODE (2.37) with $n=3$.

To this end, we look at our integrated second-order equation (2.30) for $k=1$, assuming that $Y(y)>0$ is *strictly monotone increasing* (and hence non-oscillatory) sufficiently close to the interface at $y=y_0^+<0$:

$$Y''=-\tfrac{1}{n+3}\,|Y|^{-\frac{n}{n+1}}Y\,y \quad \text{or} \quad Y''=-\tfrac{1}{n+3}\,Y^{\frac{1}{n+1}}y \;\;\text{for}\;\; Y>0. \tag{3.1}$$

Therefore, we can rewrite our derivatives of $Y(y)$ in terms of the inverse function $y(Y)$:

$$Y'=\tfrac{\mathrm{d}Y}{\mathrm{d}y}=\tfrac{1}{y'(Y)} \quad \text{and} \quad Y''=\tfrac{\mathrm{d}}{\mathrm{d}y}\big(\tfrac{1}{y'}\big)=-\tfrac{y''}{(y')^3}.$$

So, now we can reduce our differential equation (3.1) to its equivalent integral form,

$$\begin{aligned} -\tfrac{y''}{(y')^3}=-\tfrac{1}{n+3}\,Y^{\frac{1}{n+1}}y \quad &\Longleftrightarrow \quad -\tfrac{1}{2(y')^2}=-\tfrac{1}{n+3}\int_0^Y y(s)s^{\frac{1}{n+1}}\,\mathrm{d}s \\ \Longleftrightarrow \quad & y(Y)=y_0+\int_0^Y\sqrt{\tfrac{n+3}{2\int_0^r y(s)s^{\frac{1}{n+1}}\,\mathrm{d}s}}\,\mathrm{d}r\equiv M(y). \end{aligned} \tag{3.2}$$

We next prove the following property of the integral operator M in (3.2):

Proposition 3.1. *For small $\delta>0$, $M(y)$ is a contraction in $C[0,\delta]$, with the sup-norm, and therefore admits a unique fixed point $y(Y)>0$ on $(0,\delta)$ giving the unique positive solution of the ODE (3.1) on $(y_0, y_0+\varepsilon)$ with some sufficiently small $\varepsilon=\varepsilon(\delta)>0$.*

Proof. We need to show that, for $M(y)$ to be a contraction, in the C-metric,

$$\|M(\zeta_2)-M(\zeta_1)\|<\mu\|\zeta_2-\zeta_1\|, \tag{3.3}$$

for some constant $\mu=\mu(\delta)\in(0,1)$. It is easy to see that $M: Z_\delta\to Z_\delta$, for the space Z_δ of continuous functions, $Z_\delta=\{\zeta(Y)\in C[0,\delta],\ \zeta(0)=y_0\}$, with the sup-norm:

$$\|\zeta\|:=\sup\nolimits_{Y\in(0,\delta)}|\zeta(Y)|.$$

Now, take arbitrary $\zeta_1(Y), \zeta_2(Y) \in Z_\delta$. Then from (3.2) we have that

$$\|M(\zeta_2) - M(\zeta_1)\| = \sqrt{\tfrac{n+3}{2}} \int_0^Y \left\| \left(\int \zeta_2(s) s^{\frac{1}{n+1}} \, \mathrm{d}s \right)^{-\frac{1}{2}} - \left(\int \zeta_1(s) s^{\frac{1}{n+1}} \, \mathrm{d}s \right)^{-\frac{1}{2}} \right\| \mathrm{d}r,$$

where we use the simplified notation for the integral $\int$, without any limits of integration, to mean $\int_0^r$. This equality can now be written as

$$\|M(\zeta_2) - M(\zeta_1)\| = \sqrt{\tfrac{n+3}{2}} \int_0^Y \left\| \frac{\left(\int \zeta_2(s) s^{\frac{1}{n+1}} \, \mathrm{d}s \right)^{-\frac{1}{2}} - \left(\int \zeta_1(s) s^{\frac{1}{n+1}} \, \mathrm{d}s \right)^{-\frac{1}{2}}}{\left(\int \zeta_2(s) s^{\frac{1}{n+1}} \, \mathrm{d}s \right)^{-\frac{1}{2}} + \left(\int \zeta_1(s) s^{\frac{1}{n+1}} \, \mathrm{d}s \right)^{-\frac{1}{2}}} \right\| \mathrm{d}r.$$

Denoting the exponent $\nu = \frac{1}{n+1}$, we see that

$$\|M(\zeta_2) - M(\zeta_1)\| \le \sqrt{\tfrac{n+3}{2}} \int_0^Y \left\| \frac{\int [\zeta_2(s) - \zeta_1(s)] s^\nu \, \mathrm{d}s}{\int \zeta_2(s) s^\nu \, \mathrm{d}s \int \zeta_1(s) s^\nu \, \mathrm{d}s \left[\left(\int \zeta_2(s) s^\nu \, \mathrm{d}s \right)^{-\frac{1}{2}} + \left(\int \zeta_1(s) s^\nu \, \mathrm{d}s \right)^{-\frac{1}{2}} \right]} \right\| \mathrm{d}r.$$

Since we deal with sufficiently small values of Y, so that always $\zeta_{1,2}(s) \approx y_0$, it is easy to estimate in the denominator to get that

$$\begin{aligned} \|M(\zeta_2) - M(\zeta_1)\| &\le \mu_0 \int_0^Y \left\| \frac{\|\zeta_2 - \zeta_1\| \int s^\nu \, \mathrm{d}s}{\int s^\nu \, \mathrm{d}s \int s^\nu \, \mathrm{d}s \left[\left(\int s^\nu \, \mathrm{d}s \right)^{-\frac{1}{2}} + \left(\int s^\nu \, \mathrm{d}s \right)^{-\frac{1}{2}} \right]} \right\| \mathrm{d}r \\ &\le \mu_0 \|\zeta_2 - \zeta_1\| \int_0^Y \frac{r^{\frac{n+2}{n+1}}}{r^{\frac{n+2}{n+1}} r^{\frac{n+2}{n+1}} r^{-\frac{n+2}{2(n+1)}}} \, \mathrm{d}r \le \mu_0 \|\zeta_2 - \zeta_1\| \int_0^Y r^{-\frac{n+2}{2(n+1)}} \, \mathrm{d}r \le \mu_0 \|\zeta_2 - \zeta_1\| \, Y^{\frac{n}{2(n+1)}}. \end{aligned}$$

Here, μ_0 is a constant dependent on n and y_0. Since we take $\frac{1}{2}|y_0| \le y \le |y_0|$, then we have that $|Y| < 1$, and so, fixing $Y \in [0, Y_0]$, with $\mu = \mu_0 |Y_0| < 1$, we have that (3.3) holds true. Hence by Banach's Fixed Point Theorem [1, p. 39], $M(y)$ has a unique fixed point in Z_δ. $\square$

Similarly, we prove local existence and uniqueness of a positive solution for $l = 1, 2$.

For $k \ge 2$, the solutions are oscillatory (changing sign) close to interfaces, so that a contraction approach does not apply, and, as we have mentioned, we need to use other techniques of asymptotic analysis in both limits $y \to y_0^+$ and $y \to +\infty$ (the latter one includes $k = 1$). Such oscillatory structures near interfaces have been thoroughly studied for higher-order thin film equations; see [2, 3]. Examples of such oscillatory patterns for various NDEs can be found in [14, Ch. 4], so we do not address these questions here anymore, and concentrate on another and more difficult limit.

3.2. **Global existence and uniqueness for $k = 1$.** We restrict our attention to first nonlinear eigenfunction Y_0 and prove the following:

Theorem 3.1. *For any $n > 0$ and a fixed interface point $y_0 < 0$, the problem* (2.30), (2.33) *admits a unique solutions $Y_0(y)$, which is infinitely oscillatory as $y \to +\infty$.*

Proof. Once the local existence and uniqueness have been established in Proposition 3.1, its unique existence on the whole interval $(y_0, +\infty)$ follows from elementary checked local extension properties for the ODE (2.30), which is shown not to admit strong singularities

("blow-ups") at any finite point. Oscillatory character of such a solution will be shown below. $\square$

3.3. **Local and global behaviour of nonlinear eigenfunctions as $y \to +\infty$.** At this moment, we do not know the behaviour of nonlinear eigenfunctions $f_l(y)$ (or $Y_l(y)$) for $y \gg 1$. Since the first eigenfunction $Y_0(y)$ is expected to converge to our linear kernel $F(y)$ as $n \to 0^+$ (the Airy function (2.35) for $k=1$), we also expect to have reasonably similar behaviour as $y \to +\infty$. Recall that the rescaled kernel $F(y) \equiv \psi_0(y)$ has the following oscillatory slow algebraic decay as $y \to +\infty$ [4, § 2.2]: for some constant $\hat{c} \in \mathbb{R}$,

$$F(y) \sim y^{-\frac{2k-1}{4k}} \cos\left(d_k y^{\frac{2k+1}{2k}} + \hat{c}\right) \quad \text{as} \quad y \to +\infty, \quad \text{where} \quad d_k = 2k\left(\tfrac{1}{2k+1}\right)^{\frac{2k+1}{2k}}. \tag{3.4}$$

Further eigenfunctions $\psi_l(y)$, given by the derivatives (2.13), have the corresponding asymptotics via differentiating (3.4), so these are much more oscillatory and unbounded for $y \gg 1$. On the other hand, the eigenvalue problem (2.11) admits polynomial solutions (cf. (2.15) for $\mathbf{B}^*$)

$$\tilde{\psi}_l(y) \sim y^l + ... \quad \text{for} \quad \tilde{\lambda}_l = \tfrac{l+1}{2k+1}, \quad l = 0, 1, 2, ... \tag{3.5}$$

Since $\tilde{\psi}_l \notin L^2_\rho$, these are not proper eigenfunctions. However, this shows that the equation (2.11) admits polynomially growing non-oscillatory solutions as $y \to +\infty$.

Due to the complicated nature of the nonlinear equations for $n > 0$, it is not that easy to predict possible behaviour of solutions as $y \to +\infty$, where we expect them to be oscillatory as in the linear case $n = 0$. We first study existence and nonexistence of non-oscillatory solutions, which mimic the polynomials (3.5). Without loss of generality, we consider the ODE (2.21) for the first nonlinear eigenfunction $Y_0(y)$.

Proposition 3.2. *For any $n > 0$, the ODE* (2.21) *for even $k = 2, 4, ...$ admits a bundle of algebraically growing solutions as $y \to +\infty$*

$$Y(y) \sim \pm A y^m, \quad \text{where} \quad m = 2k + \tfrac{n+2}{n+1} \quad \text{and} \quad A = A(n,k) \ne 0, \tag{3.6}$$

and does not admit such non-oscillatory solutions for odd $k = 1, 3, ...$.

Proof. As a formal calculus, we substitute into (2.21) the asymptotic expression from (3.6) to get by balancing the leading terms:

$$|A|^{-\frac{n}{n+1}} = (-1)^k m(m-1)...(m-2k+1)[(2k+1)+n], \tag{3.7}$$

whence the explicit expression for A and the result. $\square$

Thus, for odd k, all the admitted behaviour as $y \to +\infty$ are not of algebraic growth, and, more plausibly, are oscillatory (as well as for all even k). To show this, we separately consider the particularly interesting case $k = 1$.

Proposition 3.3. *All the solutions of the ODE (2.21) for $k = 1$ are oscillatory as $y \to +\infty$, i.e., have infinitely many sign changes in any neighbourhood of $+\infty$.*

Proof. Assume first that, for $k = 1$,

$$Y(y) \to +\infty \quad \text{as} \quad y \to +\infty.$$

Then (2.21) yields

$$Y'' = -\tfrac{1}{n+3}\, y Y^{\frac{1}{n+1}} \ll -\tfrac{y}{n+3} \quad \text{for} \quad y \gg 1 \quad \Longrightarrow \quad Y(y) \ll -\tfrac{y^3}{6(n+1)} \to -\infty, \tag{3.8}$$

whence the contradiction. Further cases are studied similarly. The oscillatory character of all the solutions follows form the ODE in (3.8), since $\operatorname{sign} Y'' = -\operatorname{sign} Y$ (cf. $Y'' = -Y$ for harmonic oscillations). $\square$

3.4. **A priori bounds for $Y(y)$ for $k = 1$: nonlinear oscillatory tail.** We continue to study oscillatory properties of nonlinear eigenfunctions $Y_l(y)$, as $y \to +\infty$, for the basic case $k = 1$. Let us fix two successive local extremum points $1 \ll y_1 < y_2$ of $Y(y)$, where $Y'(y_1) = Y'(y_2) = 0$. Let us characterize the size of oscillations of $Y(y)$ at those points.

We initially look at the equation for the first nonlinear eigenfunction, with the eigenvalue $\alpha_0(n)$, for $k = 1$. Multiplying (2.30) by Y' and integrating over (y_1, y_2) yields

$$\textstyle\int_{y_1}^{y_2} Y''Y' = -\frac{1}{n+3} \int_{y_1}^{y_2} |Y|^{-\frac{n}{n+1}} Y Y' y.$$

Simplifying this, we see that

$$\tfrac{1}{2}\left[(Y')^2\right]_{y_1}^{y_2} = -\tfrac{(n+1)}{(n+2)(n+3)} \textstyle\int_{y_1}^{y_2} (|Y|^{\frac{n+2}{n+1}})' y.$$

After integrating by parts, we obtain

$$\tfrac{1}{2}\left[(Y')^2\right]_{y_1}^{y_2} = -\tfrac{(n+1)}{(n+2)(n+3)} \left[|Y|^{\frac{n+2}{n+1}} y\right]_{y_1}^{y_2} + \tfrac{(n+1)}{(n+2)(n+3)} \textstyle\int_{y_1}^{y_2} |Y|^{\frac{n+2}{n+1}}.$$

However, since we are looking at extremum points, where $Y'(y_1) = Y'(y_2) = 0$,

$$\left[(Y')^2\right]_{y_1}^{y_2} = 0 \quad \text{and} \quad |Y(y_2)|^{\frac{n+2}{n+1}} y_2 - |Y(y_1)|^{\frac{n+2}{n+1}} y_1 = \textstyle\int_{y_1}^{y_2} |Y|^{\frac{n+2}{n+1}}\, {\mathrm d}y.$$

Since $\int_{y_1}^{y_2} |Y|^{\frac{n+2}{n+1}}\, {\mathrm d}y > 0$, we have

$$|Y(y_2)|^{\frac{n+2}{n+1}} y_2 > |Y(y_1)|^{\frac{n+2}{n+1}} y_1, \quad \text{or, on rearranging,} \quad \left(\tfrac{|Y(y_2)|}{|Y(y_1)|}\right)^{\frac{n+2}{n+1}} > \left(\tfrac{y_1}{y_2}\right). \tag{3.9}$$

This gives a lower estimate on the character of oscillations.

Let us now look at our second nonlinear eigenfunction with $\alpha_1(n)$, where our equation is given by (2.24), $k = 1$. Multiplying by Y' and integrating yield

$$\begin{aligned} & Y'Y''y - (Y')^2 + \tfrac{1}{3+2n} |Y|^{-\frac{n}{n+1}} Y Y' y^2 = 0 \\ \Longrightarrow \quad & \tfrac{1}{2}[(Y')^2]' y - (Y')^2 + \tfrac{n+1}{(n+2)(3+2n)} (|Y|^{\frac{n+2}{n+1}})' y^2 = 0. \end{aligned}$$

Integrating between y_1 and y_2 again, we have that

$$\begin{aligned} & \tfrac{1}{2}[(Y')^2 y]_{y_1}^{y_2} - \tfrac{1}{2}\textstyle\int_{y_1}^{y_2} (Y')^2 + \tfrac{n+1}{(n+2)(3+2n)} [|Y|^{\frac{n+2}{n+1}} y^2]_{y_1}^{y_2} - \tfrac{2(n+1)}{(n+2)(3+2n)} \textstyle\int_{y_1}^{y_2} |Y|^{\frac{n+2}{n+1}} y = 0 \\ \Longrightarrow \quad & -\tfrac{3}{2}\textstyle\int_{y_1}^{y_2} (Y')^2 - \tfrac{2(n+1)}{(n+2)(3+2n)} \textstyle\int_{y_1}^{y_2} |Y|^{\frac{n+2}{n+1}} y \\ & \qquad + \tfrac{n+1}{(n+2)(3+2n)} \left(|Y(y_2)|^{\frac{n+2}{n+1}} y_2^2 - |Y(y_1)|^{\frac{n+2}{n+1}} y_1^2\right) = 0. \end{aligned}$$

One can see that

$$\tfrac{3}{2}\int_{y_1}^{y_2}(Y')^2 + \tfrac{2(n+1)}{(n+2)(3+2n)}\int_{y_1}^{y_2}|Y|^{\frac{n+2}{n+1}}y > 0,$$

since $y > 0$. Hence, we have a similar estimate:

$$|Y(y_2)|^{\frac{n+2}{n+1}}y_2^2 > |Y(y_1)|^{\frac{n+2}{n+1}}y_1^2, \quad \text{or} \quad \big(\tfrac{|Y(y_2)|}{|Y(y_1)|}\big)^{\frac{n+2}{n+1}} > \big(\tfrac{y_1}{y_2}\big)^2. \tag{3.10}$$

Finally, for the third eigenfunction with $\alpha_2(n)$, using (2.26), $k = 1$ and multiplying by Y' yields

$$\begin{aligned} &Y'Y''y^2 - 2(Y')^2y + 2YY' + \tfrac{1}{3+3n}Y^{-\frac{n}{n+1}}YY'y^3 = 0 \\ \Longrightarrow \quad &\tfrac{1}{2}[(Y')^2]'y^2 - 2(Y')^2 + (Y^2)' + \tfrac{n+1}{(n+2)(3+3n)}\,(|Y|^{\frac{n+2}{n+1}})'y^3 = 0. \end{aligned}$$

Integrating over (y_1, y_2) yields

$$\begin{aligned} \tfrac{1}{2}[(Y')^2y^2]_{y_1}^{y_2} - \int_{y_1}^{y_2}(Y')^2y - 2\int_{y_1}^{y_2}(Y')^2 + [Y^2]_{y_1}^{y_2} + \tfrac{n+1}{(n+2)(3+3n)}[|Y|^{\frac{n+2}{n+1}}y^3]_{y_1}^{y_2} \\ - \tfrac{3(n+1)}{(n+2)(3+3n)}\int_{y_1}^{y_2}|Y|^{\frac{n+2}{n+1}}y^2 = 0. \end{aligned}$$

It then follows that

$$\big(\tfrac{|Y(y_2)|}{|Y(y_1)|}\big)^{\frac{n+2}{n+1}} > \big(\tfrac{y_1}{y_2}\big)^3, \quad \text{and, for any } l, \quad \big(\tfrac{|Y(y_2)|}{|Y(y_1)|}\big)^{\frac{n+2}{n+1}} > \big(\tfrac{y_1}{y_2}\big)^{l+1}. \tag{3.11}$$

Thus, the three estimates (3.9), (3.10), and (3.11) characterize the behaviour of the nonlinear oscillatory tail for $k = 1$, which can be compared with numerical evidence in Section 2.5.

3.5. **More on oscillatory structure and periodicity.** As mentioned before, at least for small $n > 0$, we expect that nonlinear eigenfunctions exhibit, as $y \to +\infty$, a behaviour which is structurally similar to the linear kernel and its derivatives. We therefore expect to have a special oscillatory behaviour for $y \gg 1$, as we have already seen before. Hence, we look to describe this oscillatory structure and, as a first natural attempt, we will try to find if these oscillations are given by *periodic functions*. Let us introduce the *oscillatory component* $\phi(s)$ such that, as $y \to +\infty$,

$$Y(y) = y^\gamma\phi(s), \quad \text{where} \quad s = \ln y, \tag{3.12}$$

for some power $\gamma \in \mathbb{R}$. Here the term y^γ gives the rate of any growth/decay of the oscillations and may be compared to the controlling factor $y^{-\frac{2k-1}{4k}}$ found in the linear asymptotics (3.4) for $n = 0$.

Let us begin with the simpler case $k = 1$, where substituting into (3.1), we obtain

$$(y^\gamma\phi)'' + \tfrac{1}{n+3}\,y^{1+\frac{\gamma}{n+1}}|\phi|^{-\frac{n}{n+1}}\phi = 0. \tag{3.13}$$

Expanding this expression and equating powers of y, we find that

$$\gamma - 2 = 1 + \tfrac{\gamma}{n+1} \quad \Longrightarrow \quad \gamma = \tfrac{3(n+1)}{n}. \tag{3.14}$$

This gives us a second-order ODE for the oscillatory component:

$$P_2(\phi) \equiv \phi'' + (2\gamma - 1)\phi' + \gamma(\gamma - 1)\phi = -\tfrac{1}{n+3}\,|\phi|^{-\frac{n}{n+1}}\phi \quad \text{in} \quad \mathbb{R}. \tag{3.15}$$

A most typical orbit describing oscillations should be a periodic orbit of (3.15). However, since $\gamma > 0$ in (3.14), the behaviour (3.12) describes unbounded oscillations as $y \to +\infty$, which are not acceptable for a source-type solution; cf. the linear decaying one (3.4) for $n = 0$. Hence, we conclude that oscillatory behaviour as $y \to +\infty$ is not given by periodic oscillatory components $\phi(\ln y)$ as in (3.12).

Similarly, we arrive at a contradiction, applying (3.12) to the general equation (2.29):

$$(-1)^k D_y^{2k+1}(y^\gamma \phi) = \tfrac{1}{(2k+1)+(l+1)n}\, y(y^{\frac{\gamma}{n+1}}|\phi|^{-\frac{n}{n+1}}\phi)' + \tfrac{l+1}{(2k+1)+(l+1)n}\, y^{\frac{\gamma}{n+1}}|\phi|^{-\frac{n}{n+1}}\phi.$$

Balancing the polynomial terms yields

$$\gamma - (2k+1) = \tfrac{\gamma}{n+1} \quad \Longrightarrow \quad \gamma = \tfrac{(2k+1)(n+1)}{n} > 0. \tag{3.16}$$

This leaves us with an ODE of the order $2k+1$, for $\phi(s)$:

$$(-1)^k P_{2k+1}(\phi) = \tfrac{1}{(2k+1)+(l+1)n}\,(|\phi|^{-\frac{n}{n+1}}\phi)' + \tfrac{1}{n}\,|\phi|^{-\frac{n}{n+1}}\phi. \tag{3.17}$$

Here, $P_{2k+1}(\phi)$ is a polynomial operator on ϕ, induced by the term $D_y^{2k+1}(y^\gamma\phi)$. For the case $k = 1$, the polynomial operator is given by

$$P_3(\phi) = \phi''' + \tfrac{3(2n+3)}{n}\,\phi'' + \tfrac{9n^2+7n+27}{n}\,\phi' + \tfrac{3(n+1)(2n+3)(n+3)}{n^3}\,\phi.$$

In general, $P_{2k+1}(\phi)$ is defined by the recursion

$$\begin{aligned} P_{2k+1}(\phi) = {} & \tfrac{\mathrm{d}^2}{\mathrm{d}s^2}P_{2k-1}(\phi) + (\gamma - 2k)\tfrac{\mathrm{d}}{\mathrm{d}s}P_{2k-1}(\phi) + (\gamma - 2k + 1)\tfrac{\mathrm{d}}{\mathrm{d}s}P_{2k-1}(\phi) \\ & +(\gamma - 2k)(\gamma - 2k + 1)P_{2k-1}(\phi). \end{aligned}$$

As usual, periodic solutions of (3.17) are at most simple oscillatory components. However, as before, since $\gamma > 0$ in (3.16), oscillatory structures of the form (3.12), for any $k \ge 2$, are not applicable, at least for the first nonlinear eigenfunction $Y_0(y)$, which is assumed to be integrable as $y \to +\infty$.

For higher-order nonlinear eigenfunctions $Y_l(y)$, with $l \ge 1$, proving existence of periodic solutions of the ODE (3.17) is the first step in understanding the oscillatory behaviour. This is a difficult mathematical problem, which nevertheless can be solved for orders of k that are not too large. We refer to [2, 3, 11] for key references and recent results on existence-uniqueness of periodic solutions of even-order ODEs such as (3.17), which occur in parabolic thin film theory. We also refer to [7, § 4] for existence results of periodic orbits for oscillatory solutions for the PME–4 (2.1), $m = 2$.

Overall, we rule out the "periodic" structures (3.12) as $y \to +\infty$ for the first nonlinear eigenfunction $Y_0(y)$, which is expected to have an oscillatory decay and be integrable (not in the absolute sense, i.e., it is not measurable there). Therefore, in this case, the oscillatory behaviour may be more complicated and corresponds to not that easy "nonlinear focus", which we are going to catch using numerical methods. For $l \ge 1$, such a behaviour (3.12) with almost periodic oscillatory components $\phi(\ln y)$ is still plausible (but remains rather suspicious).

3.6. **Branching of nonlinear eigenfunctions at $n = 0$.** As we have promised, we now apply another classic idea to trace out the behaviour of all the nonlinear eigenfunctions for small $n > 0$. Namely, we are going to show that there exists branching at $n = 0^+$ of solutions with respect to the parameter n. In other words, we show that, as $n \to 0$, there exists certain convergence to solutions (driven by the eigenfunctions of the linear operator $\mathbf{B}$ in (1.12)) of the the LDE (1.13).

To this end, let us look at the general ODE given by (2.5). We first expand $|f|^n$ to formally get

$$|f|^n = 1 + n \ln|f| + O(n^2). \tag{3.18}$$

This is pointwise and uniformly true in any bounded positivity subset $\{|f| \geq \delta_0 > 0\}$. However, we are not at this moment going to discuss a rigorous functional meaning of this expansion for changing sign functions $f(y)$ defined in the whole $\mathbb{R}$. Note that (3.18) can then be understood in a weak sense, which may be sufficed for passing to the limit in the equivalent integral equations; see [2, § 7.6] for asymptotic details.

Thus, using the formal expansion (3.18), (2.5) reduces to

$$(-1)^{k+1} D_y^{2k+1}\big[(1 + n \ln|f|)f\big] + \tfrac{1-\alpha n}{2k+1}\, f' y + \alpha f + O(n^2) = 0.$$

Expanding coefficients for small $n > 0$ yields

$$(\mathbf{B} - \lambda_l I) f + (-1)^{k+1} D_y^{2k+1}(n \ln|f| f) + (\alpha - \tfrac{l+1}{2k+1})\, f - \tfrac{\alpha n}{2k+1}\, f' y + O(n^2) = 0, \tag{3.19}$$

where $\mathbf{B}$ is the linear operator (1.12) and $\lambda_l = -\frac{l}{2k+1}$ is its $(l+1)$th eigenvalue.

For $l < 2k+1$, we can find our eigenvalues $\alpha_l(n)$ explicitly as in (2.27), so that

$$\alpha_l(n) = \tfrac{l+1}{(2k+1)+n(l+1)} = \tfrac{l+1}{2k+1}\left[1 + \tfrac{n(l+1)}{2k+1}\right]^{-1} = \tfrac{l+1}{2k+1}\left[1 - \tfrac{n(l+1)}{2k+1}\right] + O(n^2).$$

Then (3.19) reduces to

$$(\mathbf{B} - \lambda_l I) f + (-1)^{k+1} D_y^{2k+1}(n \ln|f| f) - \tfrac{n(l+1)^2}{(2k+1)^2}\, f - \tfrac{n(l+1)}{(2k+1)^2}\, f' y + O(n^2) = 0.$$

Hence using the Lyapunov–Schmidt method [23] by setting

$$f = \psi_l + n\phi_l + O(n^2), \tag{3.20}$$

we obtain, within the order $O(n)$, the following inhomogeneous equation:

$$(\mathbf{B} - \lambda_l I)\phi_l = (-1)^k D_y^{2k+1}(\ln|\psi_l|\psi_l) + \tfrac{(l+1)^2}{(2k+1)^2}\, \psi_l + \tfrac{(l+1)}{(2k+1)^2}\, \psi_l' y \equiv h. \tag{3.21}$$

Using Hermitian spectral theory for $\mathbf{B}$ and completeness-closure of the eigenfunctions subset $\Phi = \{\psi_l\}_{l \geq 0}$ [4, § 4], for the unique solvability of (3.21) for ϕ_l, it now remains to demand that the right-hand side h is orthogonal to ψ_l^*, i.e.,

$$\langle h, \psi_l^* \rangle_* = 0. \tag{3.22}$$

Here we have to use the corresponding indefinite metric $\langle \cdot, \cdot \rangle_*$, in which the pair $\{\mathbf{B}, \mathbf{B}^*\}$ comprises the operator $\mathbf{B}$ and its adjoint (this metric can be reduced to the standard dual L^2-one, [4, § 5]). Here, (3.22) is known as a scalar bifurcation equation in the classic Lyapunov-Schmidt method [23].

We then use the adjoint polynomial eigenfunctions ψ_l^* given by (2.15). Then, for $l < 2k+1$, we have that the generalized Hermite polynomials are simple [4, § 5],

$$\psi_l^*(y) = \tfrac{1}{\sqrt{l!}}\, y^l \quad (0 \le l < 2k+1).$$

Hence, for all $l < 2k+1$, (3.22) is indeed valid:

$$\begin{aligned}\langle h, \psi_l^* \rangle_* &= \tfrac{1}{\sqrt{l!}} \int \big[(-1)^k D_y^{2k+1} (\ln |\psi_l| \psi_l) y^l + (-1)^l \tfrac{(l+1)^2}{(2k+1)^2}\, \psi_l y^l + (-1)^l \tfrac{(l+1)}{(2k+1)^2}\, \psi_l' y^{l+1} \big] \mathrm{d}y \\ &= \tfrac{1}{\sqrt{l!}} \int \big[(-1)^k D_y^{2k+1} (\ln |\psi_l| \psi_l) y^l + (-1)^l \tfrac{(l+1)}{(2k+1)^2}\, (\psi_l y^{l+1})' \big] \, \mathrm{d}y = 0.\end{aligned}$$

Recall that, for $l \ge 2k+1$, we do not know nonlinear eigenvalues $\alpha_l(n)$ explicitly. In this case, we expand $\alpha_l(n)$ as follows:

$$\alpha_l(n) = \alpha_0 + \alpha_1 n + O(n^2), \tag{3.23}$$

where $\alpha_0 = \alpha_l(0)$ in (2.12) comes from linear Hermitian theory, and α_1 is a new unknown. As before, we use (3.19) and now we substitute (3.23), as well as (3.20), to obtain

$$\begin{aligned}n(\mathbf{B} - \lambda_l I)\phi_l =&(-1)^k D_y^{2k+1} (n \ln |\psi_l| \psi_l) + \big(\alpha_0 + n\alpha_1 - \tfrac{l+1}{2k+1} \big)\, \psi_l \\ &+ n\big(\alpha_0 - \tfrac{l+1}{2k+1} \big)\, \phi_l - \tfrac{n\alpha_0}{2k+1}\, \psi_l' y + O(n^2) = 0.\end{aligned}$$

Equating as usual the terms of the order $O(n)$, we can find the value of α_0, with

$$\begin{aligned}&\alpha_0 - \tfrac{l+1}{2k+1} = 0, \quad \text{and} \quad n(\mathbf{B} - \lambda_l I)\phi_l \\ &= (-1)^k D_y^{2k+1} (n \ln |\psi_l| \psi_l) + n\alpha_1 \psi_l + n\big(\alpha_0 - \tfrac{l+1}{2k+1} \big)\, \phi_l - \tfrac{n\alpha_0}{2k+1}\, \psi_l' y.\end{aligned} \tag{3.24}$$

Hence, substituting into (3.24) and passing to the limit $n \to 0^+$ yield

$$(\mathbf{B} - \lambda_l I)\phi_l = (-1)^k D_y^{2k+1} (\ln |\psi_l| \psi_l) + \alpha_1 \psi_l - \tfrac{l+1}{(2k+1)^2}\, \psi_l' y \equiv h.$$

Then, the orthogonality condition (3.22) becomes an algebraic equation for α_1 in (3.23). Namely, taking the inner product with ψ_l^* and noting that $\langle h, \psi_l^* \rangle_* = 0$, $\langle \psi_l, \psi_l^* \rangle_* = 1$ yield

$$\alpha_1 = -\langle (-1)^k D_y^{2k+1} (\ln |\psi_l| \psi_l) - \tfrac{l+1}{(2k+1)^2}\, \psi_l' y, \psi_l^* \rangle_*. \tag{3.25}$$

Thus, this uniquely defines the second coefficient α_1 in the expansion (3.23) and then, as usual, (3.25) gives a unique function ψ_l in the eigenfunction expansion (3.20), etc.

In the analytic or even finite regularity cases, solvability conditions and existence of expansions such as (3.23) usually rigorously justify the actual presence of branching. Our case is more delicate in view of the "weakness" of the expansion (3.18). However, for the variable $Y = |f|^n f$, the expansion (3.18) is easier to justify, especially now the equation becomes semilinear and can be reduced to an integral equation with compact Hammerstein–Uryson-type operators. Therefore, in the present case, a rather full justification of the n-branching method, though being rather technical, is doable and does not represent a principally non-solvable problem of nonlinear integral operator theory.

4. The nonlinear limit $n \to \infty$: $k = 1$ and $l = 0$

While we dealt before with a "homotopy path" construction of nonlinear eigenfunctions in the limit of small $n \to 0^+$, we now consider the opposite "highly nonlinear" limit $n \to +\infty$, which also helps to understand properties of the nonlinear eigenvalue problem.

4.1. **Reducing to an algebraic problem.** As we have seen in Section 3.6, as $n \to 0$, there appears a direct connection of all the nonlinear eigenfunctions with the linear ones associated with the LDE (1.13). Now, we are going to perform the opposite *highly nonlinear limit* $n \to +\infty$ for nonlinear eigenfunctions of the NDE. Rather surprisingly, it turns out that this "limit nonlinear case" admits a more profound analysis, and for $l = 0, 1, 2$ we are able to tackle the nonlinear eigenvalue problems by a simpler geometric-algebraic, allowing us to obtain a number of analytical and explicit expressions.

Consider the ODE (2.21), with $k = 1$ and where $l = 0$, i.e., (2.30). Since we are dealing with $n \gg 1$, it is necessary to scale out any coefficients containing large n's. In order to do this, we let

$$Y(y) = C\,\tilde{Y}(y), \quad \text{where} \quad C = C(n) > 0 \quad \text{is a constant.} \tag{4.1}$$

Substituting (4.1) into (2.30) yields:

$$C\tilde{Y}'' = -\tfrac{1}{n+3}\,|C|^{-\frac{n}{n+1}}C|\tilde{Y}|^{-\frac{n}{n+1}}\tilde{Y}y \quad \Longrightarrow \quad C = (n+3)^{-\frac{n+1}{n}}, \tag{4.2}$$

so that, on scaling out such a $C(n)$, we obtain the rescaled ODE

$$\tilde{Y}'' = -|\tilde{Y}|^{-\frac{n}{n+1}}\tilde{Y}y. \tag{4.3}$$

Here we can pass to the limit $n \to +\infty$, since the only n-dependent exponent satisfies $-\frac{n}{n+1} \to -1$. At $n = +\infty$, the ODE becomes simpler and contains a bounded discontinuous nonlinearity:

$$\boxed{\mathbf{B}_\infty(\tilde{Y}) \equiv \tilde{Y}'' + \operatorname{sign}\tilde{Y}\,y = 0 \quad \text{in} \quad \mathbb{R}.} \tag{4.4}$$

Of course, passing to the limit to arrive at (4.4) might be a delicate mathematical problem. A potentially dangerous situation occurs in those subsets, where Y vanishes. However, if both $Y(y)$ and $\tilde{Y}(y)$ have a.a. zeros transversal in the natural sense, then passage to the limit is straightforward. A sufficient "transversality" of zeros of the limit function $\tilde{Y}(y)$ can be checked *a posteriori*, after completing our algebraic construction.

Solving "almost linear" ODE (4.4), we find expressions dependent on the sign of $\tilde{Y}$:

$$\begin{cases} \tilde{Y} > 0: \quad \tilde{Y}_+(y) = -\frac{1}{6}\,y^3 + c_1 y + c_2, \\ \tilde{Y} < 0: \quad \tilde{Y}_-(y) = \frac{1}{6}\,y^3 + d_1 y + d_2. \end{cases} \tag{4.5}$$

Here c_1, c_2, d_1, d_2 are all constants, not necessarily positive ones. Knowing the conditions of continuity for the function $\tilde{Y}(y)$ and $\tilde{Y}'(y)$, we must have that all one-sided limits coincide, i.e.,

$$\tilde{Y}_+ = \tilde{Y}_- \quad \text{and} \quad \tilde{Y}_+' = \tilde{Y}_-', \quad \text{at any zero, where} \quad \tilde{Y} = 0. \tag{4.6}$$

Let the points $\{y = y_i\}_{i \ge 0}$ be successive zeros, i.e., $\tilde{Y}(y_i) = 0$, for $i = 0, 1, 2, \dots$. Hence, for $\tilde{Y}_+(y_i)$ given in (4.5), for a fixed isolated zero with an $i \ge 1$ (as usual, $y_0 < 0$ corresponds to the left-hand interface),

$$\tilde{Y}_+(y_i) = -\tfrac{1}{6}\, y_i^3 + c_{1i} y_i + c_{2i} = 0.$$

We now rearrange this to find one of the unknown parameters, in terms of y_i and the parameter c_{2i}, such that $c_{1i} = \frac{1}{6}\, y_i^2 - \frac{c_{2i}}{y_i}$. We also have that $\tilde{Y}_+'(y_i) = \tilde{Y}_-'(y_i)$, hence

$$-\tfrac{1}{2}\, y_i^2 + c_{1i} = \tfrac{1}{2}\, y_i^2 + d_{1i}.$$

From this, we find a second parameter in terms of y_i and c_{2i}, where $d_{1i} = -\frac{5}{6}\, y_i^2 - \frac{c_{2i}}{y_i}$. Now, we see that from $\tilde{Y}_+ = \tilde{Y}_-$,

$$c_{1i} y_i + d_{1i} y_i + c_{2i} + d_{2i} = 0.$$

So, substituting in known values, we have our third parameter d_{2i} given by $d_{2i} = \frac{2}{3} y_i^3 + c_{2i}$.

We now see that, after substituting values for c_{1i}, d_{1i}, and d_{2i}, (4.5) can be written as

$$\begin{cases} \tilde{Y} > 0: \quad \tilde{Y}_+(y) = -\frac{1}{6}\, y^3 + (\frac{1}{6}\, y_i^2 - \frac{c_{2i}}{y_i})y + c_{2i}, \\ \tilde{Y} < 0: \quad \tilde{Y}_-(y) = \frac{1}{6}\, y^3 - (\frac{5}{6}\, y_i^2 + \frac{c_{2i}}{y_i})y + \frac{2}{3}\, y_i^3 + c_{2i}. \end{cases} \tag{4.7}$$

From the above, it is noted that $y_i \ne 0$, for any i, unless $c_{2i} = 0$.

4.2. **Existence, uniqueness, and zero properties of $\tilde{Y}_0(y)$.** We now resolve the algebraic system to get a rather complete description of some important properties of the solution $\tilde{Y}_0(y)$ obtained by such a "geometric approach. Recall first the scaling invariance of the ODE (4.4):

$$\tilde{Y}(y) \;\text{ is a solution } \quad \Longrightarrow \quad \pm\, a^3 \tilde{Y}\big(\tfrac{y}{a}\big) \;\text{ is a solution for any } a > 0. \tag{4.8}$$

Therefore, choosing the interface at $y_0 = -1$, we put two conditions there

$$\tilde{Y}(-1) = \tilde{Y}'(-1) = 0, \tag{4.9}$$

and prove the following:

Theorem 4.1. *The problem* (4.4), (4.9) *admits a unique nontrivial solution* $\tilde{Y}_0(y)$, *which has transversal zeros* $\{y_i\}_{i \ge 1}$ *such that*

$$\begin{aligned} &y_0 = -1, \quad y_1 = 2|y_0| = 2, \quad y_2 = 3\sqrt{2} - 1 = 3.2426..., \quad \textit{and} \\ &y_{i+1} = \frac{\sqrt{17 y_i^2 - 4 y_i y_{i-1} - 4 y_{i-1}^2} - y_i}{2} \quad \textit{for any} \quad i \ge 2. \end{aligned} \tag{4.10}$$

Proof. As we have promised, we construct such a solution using pure algebraic manipulations. Let us begin with the first interval of positivity $(y_0 = -1, y_1)$, where according to (4.9), the solution reads

$$\tilde{Y}_+(y) = -\tfrac{1}{6}\, (y+1)^2 \big(y - y_1\big). \tag{4.11}$$

Since (4.5) implies no quadratic term $\sim y^2$ in the cubic polynomial, this uniquely gives $y_1 = -2y_0 = 2$, and hence

$$\tilde{Y}_+(y) = -\tfrac{1}{6}\,(y+1)^2\big(y-2\big) > 0 \quad \text{on} \quad (-1,2). \tag{4.12}$$

On the next interval $(y_1 = 2, y_2)$, the negative solution takes the form:

$$\tilde{Y}_-(y) = \tfrac{1}{6}\,(y-2)(y-y_2)\big(y+c_1\big) < 0. \tag{4.13}$$

Similarly to the above, we then conclude that

$$c_1 = 2 + y_2. \tag{4.14}$$

Then matching of the first derivatives (4.6) at $y = y_1 = 2$ implies

$$-\tfrac{1}{6}\,3^2 = \tfrac{1}{6}\,(2-y_2)(4+y_2) \quad \Longrightarrow \quad y_2^2 + 2y_2 - 17 = 0 \quad \Longrightarrow \quad y_2 = 3\sqrt{2} - 1, \tag{4.15}$$

and this procedure can be continued.

Consider an arbitrary interval (y_{i-1}, y_i) of positivity (or negativity), $i \ge 2$, with

$$\tilde{Y}_+(y) = (y-y_{i-1})(y-y_i)\big(-\tfrac{1}{6}\,y + c_{i-1}\big), \quad \text{where} \quad c_{i-1} = -\tfrac{1}{6}\,(y_i + y_{i-1}). \tag{4.16}$$

Similarly, on the next negativity (or resp. positivity) interval (y_i, y_{i+1}),

$$\tilde{Y}_-(y) = (y-y_i)(y-y_{i+1})\big(\tfrac{1}{6}\,y + c_i\big), \quad \text{where} \quad c_i = \tfrac{1}{6}\,(y_i + y_{i+1}). \tag{4.17}$$

Therefore, the matching at $y = y_i$ yields

$$\begin{aligned} &(y_i - y_{i-1})\big(-\tfrac{1}{6}\,y_i - \tfrac{1}{6}\,(y_i + y_{i-1})\big) = (y_i - y_{i+1})\big(\tfrac{1}{6}\,y_i + \tfrac{1}{6}\,(y_i + y_{i+1})\big) \\ &\qquad \Longrightarrow \quad y_{i+1}^2 + y_i y_{i+1} - 4y_i^2 + y_i y_{i-1} + y_{i-1}^2 = 0, \end{aligned} \tag{4.18}$$

whence the final result in (4.10). Thus, the unique solution can be extended indefinitely for arbitrary $y \gg 1$ and is infinitely oscillatory as $y \to +\infty$. $\square$

Remark. The quadratic equation in (4.18) reduces to a 2D *linear discrete equation*:

$$\alpha_{i,j} \equiv y_i y_j \quad \Longrightarrow \quad \alpha_{i+1,i+1} + \alpha_{i,i+1} - 4\alpha_{i,i} + \alpha_{i,i-1} + \alpha_{i-1,i-1} = 0 \quad \text{for} \quad i \ge 2. \tag{4.19}$$

It is not difficult to find some particular solutions:

$$\alpha_{i,j} = \mu^i \nu^j \quad \Longrightarrow \quad \mu(\mu+1)\nu^2 - 4\mu\nu + \mu + 1 \quad \Longrightarrow \quad \nu_\pm(\mu) = \tfrac{2\mu \pm (\mu-1)\sqrt{-\mu}}{\mu(\mu+1)}. \tag{4.20}$$

Therefore, denoting by $\mathcal{M}$ a proper subset of parameters μ such that $\{\mu^i\}_{\mu \in \mathcal{M}}$ is complete/closed, the general solution of (4.20) is represented by a converging infinite series

$$\alpha_{i,j} = \textstyle\sum_{\mu \in \mathcal{M}} C_\mu \mu^i \nu^j, \tag{4.21}$$

where $\{C_\mu\}$ are constants and $\nu = \nu(\mu)$ take values according to (4.20). Here, (4.21) is a discrete analogy of eigenfunction expansions of solutions of a linear PDE with two independent variables (x,t). A proper posing "boundary conditions" for (4.19) to specify the corresponding Sturm–Liouville problem and next initial conditions to get the corresponding eigenfunction expansion (4.21) is a difficult and uncertain problem[1], which will unlikely provide us with any useful finite explicit formulae.

[1] It seems, a suitable behaviour at infinity, as $i, j \to +\infty$, of $\alpha_{i,j}$ might include something like the "minimality" condition in (2.6), which is hard to take into account.

However, the analytic relationships such as (4.10) can promise to get extra asymptotic properties of the first rescaled eigenfunction $\tilde{Y}_0(y)$, especially the decay rate of the minimal behaviour indicated in (2.6). However, in the present case $l=0$, unlike the simpler one $l=2$ in Section 6, some computations are not expected to be easy all the way, since such algebraic relations are quadratic and hence not always explicitly solvable.

4.3. **Numerics for $n \gg 1$: $k=1$ and $l=0$.** As usual, very sharp proper numerics can help to detect further properties of $\tilde{Y}_0(y)$, and hence avoid trying to get too complicated and exhaustive results concerning the corresponding algebraic system. Moreover, which is even more important, we can also check the character of convergence of solutions as $n \to +\infty$, which, after scaling (4.1), turns out to be rather fast. To deal with the ODE (4.4), it is possible to just use a simple shooting method using the ODE solver `ode45`, to find suitable profiles and nonlinear eigenfunctions.

We use a similar shooting method as that applied for the general case of $n>0$, set out in Section 2.5. We recall that, close to the interface at some $y=y_0<0$, we look for small solutions of $\tilde{Y}(y)$ such that, as $y \to y_0^+$,

$$\begin{aligned} \tilde{Y}(y) &= C_0(y-y_0)_+^2\big(1+o(1)\big), \quad \text{where} \quad C_0=\tfrac{1}{2}\,|y_0|>0, \\ \text{and} \quad \tilde{Y}'(y) &= |y_0|(y-y_0)(1+o(1)) \quad (k=1). \end{aligned} \tag{4.22}$$

The proof of this expansion is similar and even simpler than that of Proposition 3.1.

In Figure 8, we show the general view of the first eigenfunction $\tilde{Y}_0(y)$ on the large interval $[y_0=-1,100]$. The envelope of the decaying oscillations are governed by the algebraic curve

$$L_0(y) \approx \pm 0.7\, y^{-\frac{1}{3}} \quad \text{as} \quad y \to +\infty. \tag{4.23}$$

It seems that, definitely, this decay can be seen from the above algebraic system. We have checked that, for $n=100$, the corresponding nonlinear eigenfucntion (after scaling) is practically indistinguishable.

In Figure 9, we show $\tilde{Y}_0(y)$ on a smaller interval $y \in [-1,7]$, indicating all the first zeros, which will coincide with those given by explicit algebraic expressions in (4.10).

All the computations have been performed with the enhanced accuracy and tolerances $\sim 10^{-10}$, so these are quite reliable. Let us present first 15 zeros of $\tilde{Y}_0(y)$:

$$\begin{gathered} y_0=-1,\ y_1=2|y_0|=2,\ y_3=3.2426...,\ y_4=5.0777...,\ y_5=5.8426...,\\ y_6=6.5459...,y_7=7.2021...,\ y_8=7.8207...,\ y_9=8.4083...,\ y_{10}=8.9697...,\\ y_{11}=9.5086...,\ y_{12}=10.0279...,\ y_{13}=10.5299...,\ y_{14}=10.0164...,\ ... . \end{gathered}$$

4.4. **Branching at $n=+\infty$.** Introducing the small parameter in (4.3) for $n \gg 1$,

$$\varepsilon = 1-\tfrac{n}{n+1} \to 0^+ \quad \text{as} \quad n \to +\infty,$$

and performing a standard (formal, as usual, at least in the pointwise sense) linearization yield the following problem:

$$|\tilde{Y}|^{-\frac{n}{n+1}} \equiv |\tilde{Y}|^{\varepsilon-1} = \tfrac{1+\varepsilon \ln|\tilde{Y}|+O(\varepsilon^2)}{|\tilde{Y}|} \quad \Longrightarrow \quad \mathbf{B}_\infty(\tilde{Y}) = -\text{sign}\,\tilde{Y}\,y\,\ln|\tilde{Y}|+O(\varepsilon), \tag{4.24}$$

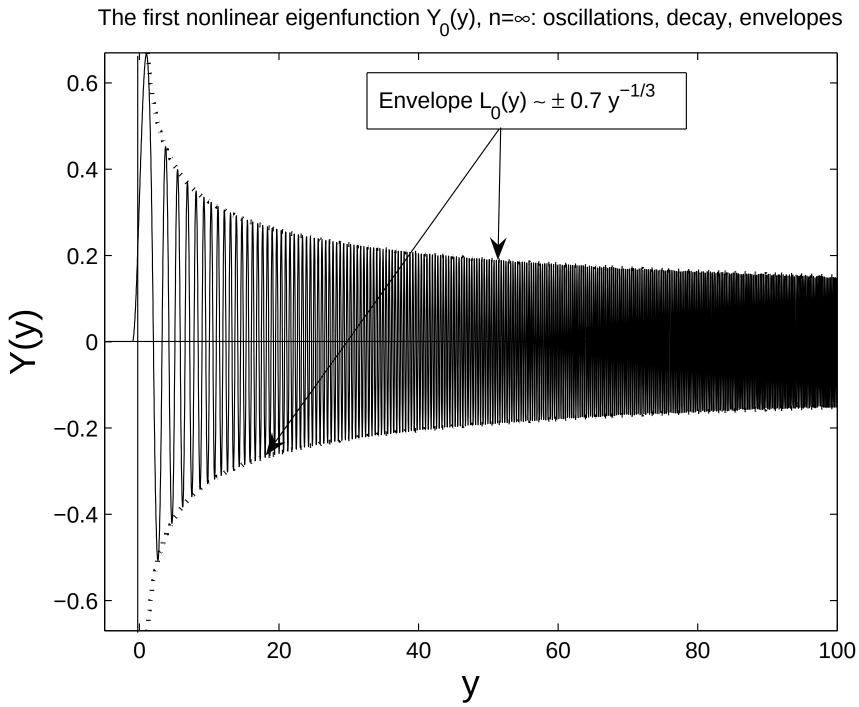

FIGURE 8. The first nonlinear eigenfunction $\tilde{Y}_0(y)$ as a solution of the problem (4.4), (4.22); $k = 1$, $l = 0$.

where $\mathbf{B}_\infty$ is the unperturbed operator in (4.4). Next, studying the branching at $\varepsilon = 0$ from the first nonlinear eigenfunction $\tilde{Y}_0$, we obtain the corresponding linear problem:

$$\tilde{Y} = \tilde{Y}_0 + \varepsilon\phi + O(\varepsilon^2) \quad \Longrightarrow \quad \mathbf{B}'_\infty(\tilde{Y}_0)\phi = h \equiv -(\mathrm{sign}\,\tilde{Y}_0)y\,\ln|\tilde{Y}_0|, \tag{4.25}$$

where the symmetric linearized operator $\mathbf{B}'_\infty(\tilde{Y}_0)$ is given by

$$\mathbf{B}'_\infty(\tilde{Y}_0) = D_y^2 + (\mathrm{sign}\,\tilde{Y}_0)'y\,I, \quad (\mathrm{sign}\,\tilde{Y}_0)' = \delta(y-y_0) + 2\sum_{(i\geq 1)}(-1)^i\delta(y-y_i). \tag{4.26}$$

Here $\{y_i\}$ are zeros of $\tilde{Y}_0(y)$ as explained in Theorem 4.1.

At this stage, one then needs proper spectral theory for a self-adjoint extension of the operator $\mathbf{B}'_\infty(\tilde{Y}_0)$. Then the branching condition reads as the orthogonality

$$h \perp \ker \mathbf{B}'_\infty(\tilde{Y}_0). \tag{4.27}$$

However, developing such a proper spectral theory faces some hard algebraic difficulties. Indeed, looking for eigenfunctions φ_k,

$$\varphi_k'' - \lambda_k\varphi_k = C_k = \mathrm{const.} \equiv -\big(y_0\varphi_k(y_0) + 2\textstyle\sum(-1)^i y_i\varphi_k(y_i)\big), \quad \varphi_k(y_0) = 0, \tag{4.28}$$

for $\lambda_k < 0$, we obtain the solution and an algebraic equation for such eigenvalues:

$$\varphi_k(y) = \sin(\sqrt{|\lambda_k|}(y-y_0)) \quad \Longrightarrow \quad \boxed{\lambda_k: \quad \sum_{(i\geq 1)}(-1)^i y_i \sin(\sqrt{|\lambda_k|}(y_i-y_0)) = 0.} \tag{4.29}$$

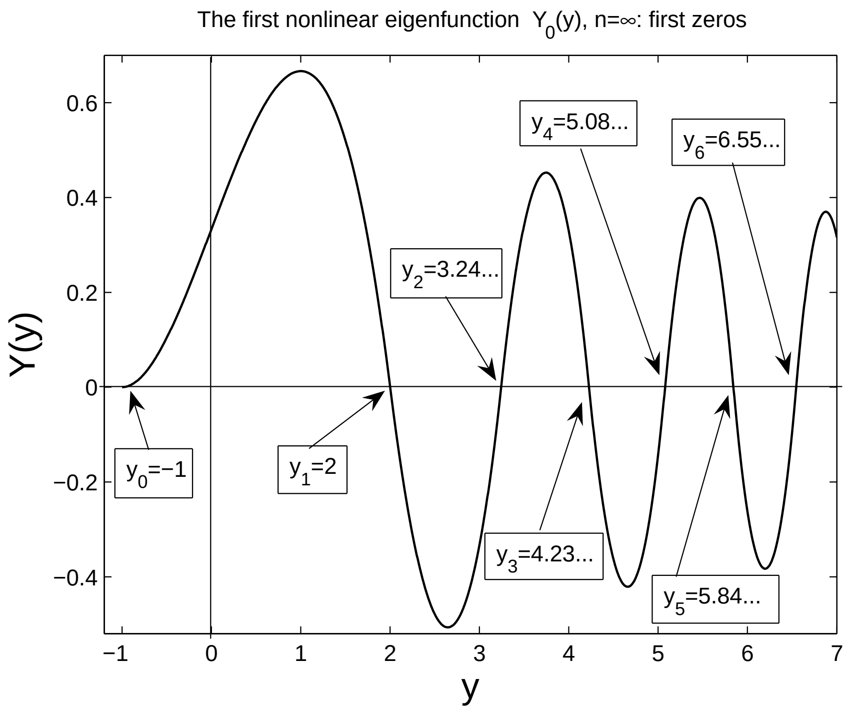

FIGURE 9. The first nonlinear eigenfunction $\tilde{Y}_0(y)$ as a solution of the problem (4.4), (4.22); $k = 1$, $l = 0$.

According to (4.29), we require that each eigenfunction $\varphi_k(y)$ should be purely oscillatory as $y \to +\infty$ about zero, meaning a "minimal" ("zero-average") behaviour or zero value in a weak sense. For $\lambda_0 = 0$, we take $\varphi_0(y) = y - y_0$ (again up to a normalization multiplier), leading to another algebraic problem, which is expected to be non-proper since the eigenfunctions is not oscillatory and not satisfying the "zero condition" at infinity.

Proving that the algebraic equation in (4.29) has a discrete family of solutions $\{\lambda_k < 0\}$ is a difficult open problem. Nevertheless, at least, this analysis shows a principal possibility to detect branching of nonlinear eigenfunctions at $n = +\infty$, where a simpler algebraic treatment is available, and more practical asymptotic and other properties of the nonlinear eigenfunctions than in Theorem 3.1, for any $n > 0$, are known.

4.5. **The higher-order case $k \geq 2$ for $l = 0$.** For the general case of the higher-order ODEs (2.21), again for $l = 0$, the equations as $n \to \infty$ are much the same. Here the scaling constant $C(n)$, for $Y = C(n)\tilde{Y}$ is given by

$$C(n) = \big[(2k+1) + n\big]^{-\frac{n+1}{n}}.$$

After scaling, this yields the ODE

$$\mathbf{B}_{\infty}^{(2k)}(\tilde{Y}) \equiv (-1)^{k+1} D_y^{2k} \tilde{Y} + \operatorname{sign} \tilde{Y}\, y = 0, \tag{4.30}$$

which deserves further study by deriving the corresponding algebraic structures. Note that, close to the interface at $y = y_0^+$, for $k \geq 2$ (the case $k = 1$ is not oscillatory, as we

have seen in Section 3), a stabilization to a *periodic oscillatory component* $\phi(s)$, with the typical structure near the interface at $y = y_0^+$,

$$\tilde{Y}(y) = (y - y_0)^{2k}(\phi(s) + o(1)), \quad \text{where} \quad s = -\ln(y - y_0) \to -\infty, \tag{4.31}$$

of solutions of (4.30) is most plausible. See [2, 3], where such oscillatory sign changing solutions such as (4.31), with a periodic component $\phi(s)$ satisfying a nonlinear higher-order ODE, have been found for the fourth- and sixth-order thin film equations.

However, explicitly solving the ODE (4.30) for any $k \ge 2$ in the positivity and negativity domains leads to much more complicated algebraic systems, which do not admit such a clear explicit resolving as for $k = 1$. Moreover, even shooting numerical methods lead to difficult and unclear results, so we do not present such an analysis here.

5. The second eigenfunction $\tilde{Y}_1(y)$ for $n = +\infty$: algebraic approach

Using the same method, the equations governing the behaviour as $n \to \infty$, relating to the second nonlinear eigenfunction $\tilde{Y}_1(y)$, can easily be found. Namely and analogously, for $l = 1$ in the ODE (2.24), with $k = 1$, the limit equation at $n = +\infty$ is given by

$$\tilde{Y}''y - \tilde{Y}' + \operatorname{sign}\tilde{Y}\, y^2 = 0. \tag{5.1}$$

This ODE (5.1) can be written in a singular Sturm–Liouville form

$$\big(\tfrac{\tilde{Y}'}{y}\big)' = -\operatorname{sign}\tilde{Y}, \tag{5.2}$$

where the weight $\rho(y) = \frac{1}{y} \not\in L^1_{\rm loc}$. Therefore, $y = 0$ is a singular inner point, where an extra condition must be posed. This is done by checking the functional weighted L^2-space corresponding to the operator in (5.2) and its available asymptotics as $y \to 0$:

$$\textstyle\int_0 \rho(y)(\tilde{Y}'(y))^2\,{\rm d}y < \infty \quad \Longrightarrow \quad \tilde{Y}'(0) = 0. \tag{5.3}$$

Next, integrating (5.2) twice yields, in the positivity and negativity domains, the following:

$$\tilde{Y}_\pm(y) = \mp\tfrac{y^3}{3} + ay^2 + b, \quad a, b, \in \mathbb{R}. \tag{5.4}$$

Note that, unlike (4.5) for $l = 0$, here the linear term $\sim y$ is absent in the cubic polynomial. The analysis of the algebraic matching system corresponding to (5.4) is similar in many places but a couple of ones, which we will concentrate upon now.

Theorem 5.1. *The problem* (5.2), (4.9) *admits a unique nontrivial solution* $\tilde{Y}_1(y)$ *with transversal zeros at* $\{y_i\}_{i \ge 1}$, *where*

$$y_0 = -1, \quad y_1 = \tfrac{1}{2}, \quad y_2 = \tfrac{5+3\sqrt{5}}{4} = 2.927050...\,, \tag{5.5}$$

and triples of zeros $\{y_{i-1}, y_i, y_{i+1}\}_{i \ge 2}$ *satisfy a cubic homogeneous algebraic equation,*

$$y_{i-1}y_{i+1}^2 + y_iy_{i+1}^2 + y_{i-1}^2y_i + y_{i-1}^2y_{i+1} = y_{i-1}y_iy_{i-1} + y_i^2y_{i+1} + y_{i-1}y_i^2 + y_i^3 \quad \textit{for} \quad i \ge 2. \tag{5.6}$$

Remark. (5.6) reduces to a 3D linear discrete equation for $\alpha_{i,j,k} = y_i y_j y_k$, but, similar to (4.18), this does not essentially help to get any finite explicit expressions for zeros $\{y_i\}$.

Proof. The first step is elementary: on $(-1, y_1)$, in view of (5.4) (cf. also (5.3)),

$$\tilde{Y}_+(y) = (y+1)^2(y-y_1)\big(-\tfrac{1}{3}\big), \quad \text{where} \quad y_1 = \tfrac{1}{2}. \tag{5.7}$$

Next, on (y_1, y_2) by (5.4),

$$\tilde{Y}_-(y) = (y-y_1)(y-y_2)\big(\tfrac{1}{3}\,y + c_1\big), \quad \text{where} \quad c_1 = \tfrac{1}{3}\,\tfrac{y_1 y_2}{y_1+y_2}, \tag{5.8}$$

so that the matching (4.6) at $y = y_1$ yields

$$(y_1+1)^2\big(-\tfrac{1}{3}\big) = (y_1-y_2)\big(\tfrac{1}{3}\,y_1 + c_1\big) \quad \Longrightarrow \quad 4y_2^2 - 10y_2 - 5 = 0, \tag{5.9}$$

whence the desired value of the root y_2 in (5.5).

Similarly, in the general case, on (y_{i-1}, y_i),

$$\tilde{Y}_-(y) = (y-y_{i-1})(y-y_i)\big(\tfrac{1}{3}\,y + c_{i-1}\big), \quad \text{where} \quad c_{i-1} = \tfrac{1}{3}\,\tfrac{y_{i-1}y_i}{y_{i-1}+y_i}, \tag{5.10}$$

and on (y_i, y_{i+1}),

$$\tilde{Y}_+(y) = (y-y_i)(y-y_{i+1})\big(-\tfrac{1}{3}\,y + c_i\big), \quad \text{where} \quad c_i = -\tfrac{1}{3}\,\tfrac{y_i y_{i+1}}{y_i+y_{i+1}}. \tag{5.11}$$

Then the standard matching via (4.6) of such $\tilde{Y}_\pm(y)$ yields the homogeneous cubic algebraic equation (5.6) for the zero triple $\{y_{i-1}, y_i, y_{i+1}\}$. $\square$

Figure 10 shows the general structure of the second eigenfunction $\tilde{Y}_1(y)$ on the large interval $[y_0 = -1, 200]$. The envelope of the decaying oscillations are governed by the algebraic curve

$$L_1(y) \approx \pm 0.89\, y^{\frac{1}{3}} \quad \text{as} \quad y \to +\infty, \tag{5.12}$$

which can be associated with the algebraic manipulations involved.

The next, Figure 11, shows first zeros of $\tilde{Y}_1(y)$ on the interval $[-1, 5]$.

In general, for all values of $k \ge 1$, we have the limit equation at $n = +\infty$ for $l = 1$ in the form:

$$(-1)^{k+1} D_y^{2k}\tilde{Y} y + (-1)^k D_y^{2k-1}\tilde{Y} + \operatorname{sign}\tilde{Y} = 0.$$

This can be integrated once, however, one cannot expect any reasonably easy algebraic manipulations leading to some explicit representation of asymptotic properties of $\tilde{Y}_1(y)$ for any $k \ge 2$.

6. The third eigenfunction $\tilde{Y}_2(y)$ for $n = +\infty$: algebraic approach

Similarly, for the third eigenfunction, where $l = 2$, we have, in the lower-order case $k = 1$, the limit ODE

$$\tilde{Y}'' y^2 - 2\tilde{Y}' y + 2\tilde{Y} + \operatorname{sign}\tilde{Y}\, y^3 = 0. \tag{6.1}$$

The corresponding Sturm–Liouville form

$$\big(\tfrac{\tilde{Y}'}{y^2}\big)' = -\tfrac{2}{y^4}\,\tilde{Y} - \tfrac{1}{y}\operatorname{sign}\tilde{Y}, \tag{6.2}$$

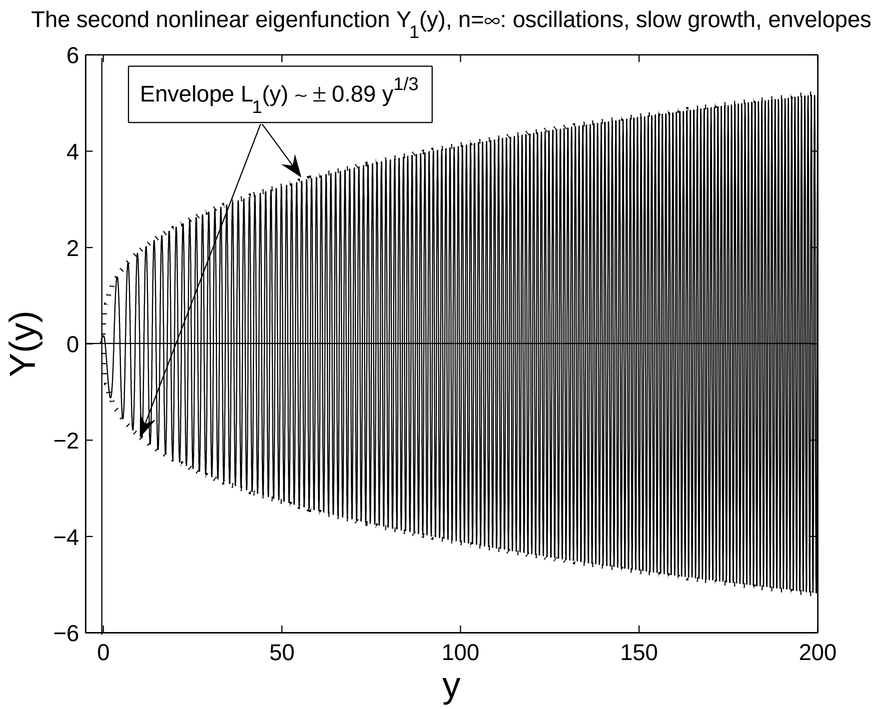

FIGURE 10. The second nonlinear eigenfunction $\tilde{Y}_1(y)$ as a solution of the problem (5.2), (4.22); $k = 1$, $l = 1$.

reveals even more singularity at $y = 0$ as it used to be in (5.2) for $\tilde{Y}_1(y)$. The necessary extra condition at $y = 0$ is presented below at (6.10).

Writing (6.1) down for $\tilde{Y}_\pm(y)$ in the form

$$Y'' = \mp y + 2\left(\tfrac{\tilde{Y}}{y}\right)' \tag{6.3}$$

and integrating twice yields

$$Y_\pm(y) = \mp \tfrac{y^3}{2} + ay^2 + by, \quad a, b \in \mathbb{R}. \tag{6.4}$$

Surprisingly, the matching at zeros $y = y_i$ by using the cubic polynomials (6.4), having $y = 0$ as a fixed zero always leads to a simpler mathematics.

Theorem 6.1. *The problem* (6.1), (4.9) *admits a unique nontrivial solution* $\tilde{Y}_1(y)$ *with transversal zeros at* $\{y_i\}_{i \ge 1}$, *where*

$$y_0 = -1, \quad y_1 = 0, \quad y_2 = 1 + \sqrt{2} = 2.4142..., \quad y_3 = 1 + 3\sqrt{2} = 5.2426..., \tag{6.5}$$

and further zeros are given by the second-order linear discrete equation: for any $i \ge 3$,

$$y_{i+1} - 2y_i + y_{i-1} = 0 \quad \Longrightarrow \quad y_i = C_1 + C_2 i, \quad \text{where} \quad C_1 = 1 - 3\sqrt{2},\ C_2 = 2\sqrt{2}. \tag{6.6}$$

In particular, the distribution of zeros is uniform:

$$y_{i+1} - y_i = 2\sqrt{2} = 2.8284... \quad \text{for all } i \ge 2. \tag{6.7}$$

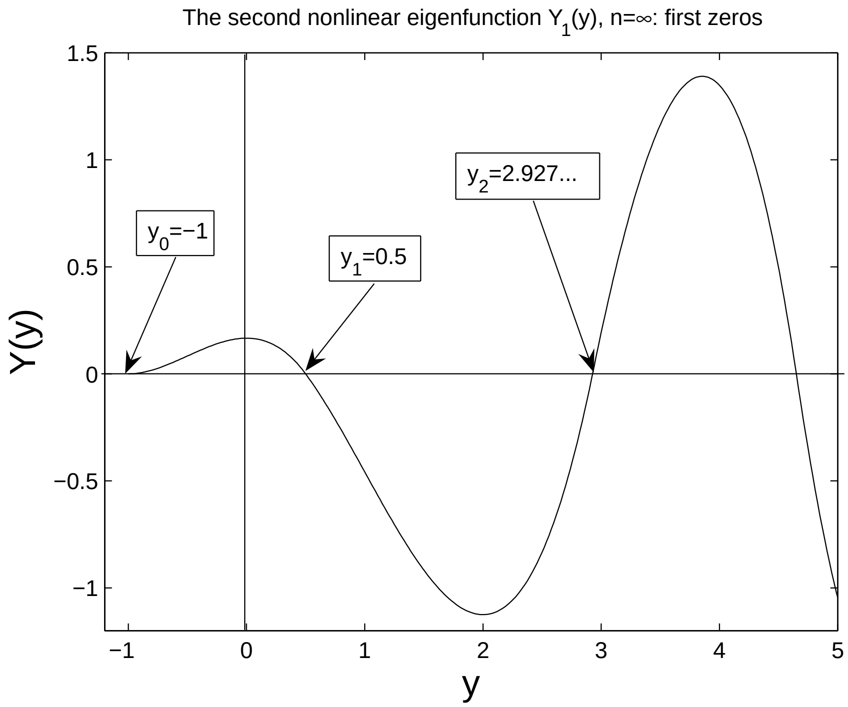

FIGURE 11. The second nonlinear eigenfunction $\tilde{Y}_1(y)$ as a solution of the problem (5.2), (4.22); $k = 1$, $l = 1$.

Proof. On $(-1, y_1)$, in view of (6.4),

$$\tilde{Y}_+(y) = y(y+1)^2\big(-\tfrac{1}{2}\big) \quad \Longrightarrow \quad y_1 = 0. \tag{6.8}$$

Next, on $(y_1 = 0, y_2)$ by (6.4),

$$\tilde{Y}_-(y) = y(y-y_2)\big(\tfrac{1}{2}\,y + c_1\big), \quad \text{where} \quad c_1 = \tfrac{1}{2y_2}. \tag{6.9}$$

Unlike the previous cases, it is not possible to find y_2 and c_1 by using the standard matching conditions (4.6). The point is that the differential operator in (6.2) is strongly singular at $y = 0$, where the weight $\rho(y) = \frac{1}{y^2} \notin L^p(-1, 1)$ for any $p \ge 1$.

It then follows from (6.4) due to the singular setting (6.2) that the two usual conditions at $y = 0$ such as the values of $\tilde{Y}(0) = 0$ and of a given "flux" $\tilde{Y}'(0)$ are not sufficient to determine a unique local solution for $y > 0$ and $y < 0$. To get a unique solution, the value of $\tilde{Y}''(0)$ should be prescribed. A further analysis shows that a proper stronger continuity condition of matching at $y = 0$ is necessary and this includes the equality of the second-order derivatives:

$$Y_+''(0^-) = Y_-''(0^+). \tag{6.10}$$

Overall, this yields the following quadratic equation for y_2:

$$Y_+''(0) = -2 = Y_-''(0) = 2c_1 - y_2, \;\; c_1 = \tfrac{1}{2y_2} \quad \Longrightarrow \quad y_2^2 - 2y_2 - 1 = 0, \tag{6.11}$$

and this uniquely defines y_2 shown in (6.5).

Next, we use (6.9) with the obtained values of y_2 and c_1 to match (now, in a standard way) with the solution representation on (y_2, y_3),

$$\tilde{Y}_+(y) = y(y-y_2)(y-y_3)\big(-\tfrac{1}{2}\big), \tag{6.12}$$

to get at $y = y_2$

$$y_2\big(\tfrac{1}{2}\,y_2 + c_1\big) = y_2(y_2-y_3)\big(-\tfrac{1}{2}\big) \quad \Longrightarrow \quad y_3 = 2y_2 + 2c_1 = 1 + 3\sqrt{2}. \tag{6.13}$$

Finally, in the general case, on (y_{i-1}, y_i) for $i \ge 3$,

$$\tilde{Y}_+(y) = y(y-y_{i-1})(y-y_i)\big(-\tfrac{1}{2}\,\big), \tag{6.14}$$

and on (y_{i1}, y_{i+1}),

$$\tilde{Y}_-(y) = y(y-y_i)(y-y_{i+1})\tfrac{1}{2}. \tag{6.15}$$

By the standard matching at $y = y_i$ via (4.6) of such $Y_\pm(y)$ yields

$$y_i(y_i-y_{i-1})\big(-\tfrac{1}{2}\big) = y_i(y_i-y_{i+1})\tfrac{1}{2}, \tag{6.16}$$

whence the linear difference relation (6.6) with some constants C_1 and C_2. These are uniquely obtained from the linear algebraic system,

$$\begin{cases} y_2 = 1+\sqrt{2} = C_1 + 2C_2, \\ y_3 = 1+3\sqrt{2} = C_1 + 3C_2 \end{cases} \quad \Longrightarrow \quad \begin{cases} C_1 = 1-3\sqrt{2}, \\ C_2 = 2\sqrt{2}, \end{cases} \tag{6.17}$$

completing the proof. $\square$

Figure 12 shows the general structure of the third eigenfunction $\tilde{Y}_2(y)$ on the large interval $[y_0 = -1, 200]$. Directly connected with (6.6), the envelope of the decaying oscillations is linear

$$L_2(y) \approx \pm y \quad \text{as} \quad y \to +\infty. \tag{6.18}$$

First zeros of $\tilde{Y}_2(y)$ on the interval $[-1, 7]$ are shown in Figure 13.

For arbitrary $k \ge 2$, the limit ODE for $\tilde{Y}_2(y)$ is

$$(-1)^{k+1}D_y^{2k}\tilde{Y}y^2 + 2(-1)^k D_y^{2k-1}\tilde{Y}y + 2(-1)^{k+1}D_y^{2k-2}\tilde{Y} + \tfrac{\tilde{Y}}{|\tilde{Y}|}\,y^3 = 0,$$

which does not admit a clear geometric-algebraic method of solution.

Finally, we recall that exotic profiles such as $\tilde{Y}_0(y)$ in Figure 8, $\tilde{Y}_1(y)$ in Figure 10 and $\tilde{Y}_2(y)$ in Figure 12 are not just some functions defining some self-similar solutions in the nonlinear limit ($n = +\infty$) equation (1.16) for $k = 1$, but these are the most stable asymptotic pattern of such a nonlinear PDE. Indeed, $\tilde{Y}_0(y)$ is the most stable and is expected to attract, as $t \to +\infty$, a.a. solutions, excluding only those with data satisfying some extra orthogonality conditions. For instance, those having zero mass (or also the zero moment for $Y_2(y)$ to play a role), so that an extra time-scaling is necessary to get convergence to the second nonlinear eigenfunction $\tilde{Y}_1(y)$, etc. Of course, these stability questions are far beyond the scope of this paper, are very difficult, and remain open for most higher-order NDEs.

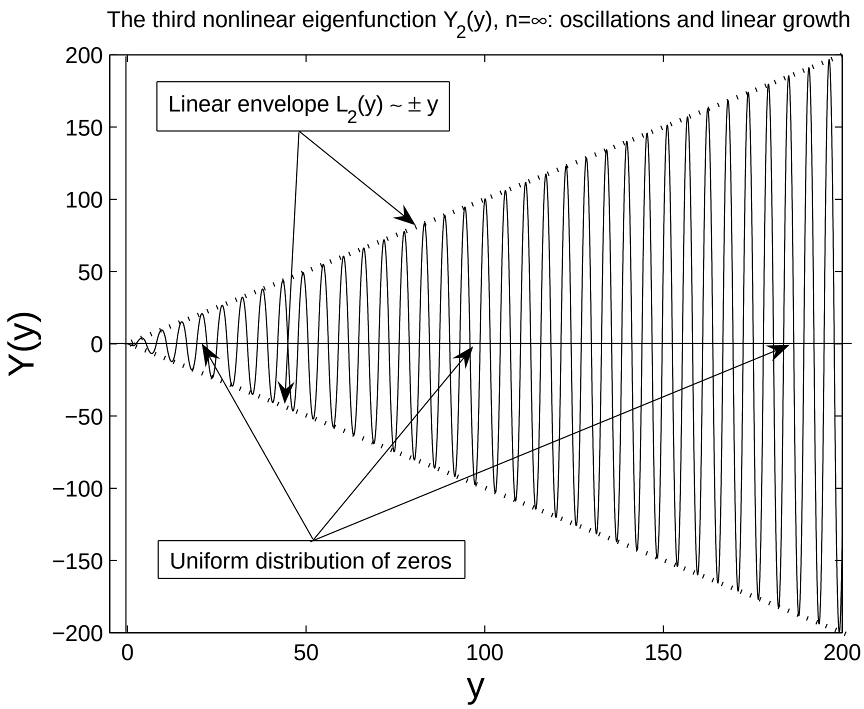

FIGURE 12. The third nonlinear eigenfunction $\tilde{Y}_2(y)$ as a solution of the problem (6.3), (4.22); $k = 1$, $l = 2$.

7. Nonlinear dispersion equation with absorption

7.1. **A full quasilinear NDE: homotopy deformation to linear PDEs.** The final natural progression, from the nonlinear model (1.11), is to go to the odd-order NDE with absorption (1.1). This links up both the nonlinear model and the semilinear one [4],

$$u_t = (-1)^{k+1} D_x^{2k+1} u - |u|^{p-1} u \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+ \quad (n = 0). \tag{7.1}$$

However, the NDE (1.1) is indeed a more difficult quasilinear equation than (7.1), with not that well understood phenomena of "nonlinear" bifurcation and branching. Therefore, we should pay here less effort towards establishing rigorously some of the key analytical aspects concerning basic similarity solutions of (1.1).

The NDE (1.1), for any $n > 0$, $p > n + 1$, after similarity scaling,

$$u_{\rm gl}(x,t) = t^{-\frac{1}{p-1}} f(y), \quad y = \tfrac{x}{t^{\beta}}, \quad \text{where} \quad \beta = \tfrac{p-(n+1)}{(p-1)(2k+1)}, \tag{7.2}$$

reduces to the ODE

$$(-1)^{k+1} D_y^{2k+1}(|f|^n f) + \tfrac{p-(n+1)}{(p-1)(2k+1)}\, f' y + \tfrac{1}{p-1}\, f - |f|^{p-1} f = 0 \quad \text{in} \quad \mathbb{R}. \tag{7.3}$$

As usual, a proper setting for (7.3) assumes a finite left-hand interface at some $y = y_0 < 0$ and an admissible oscillatory behaviour as $y \to +\infty$, which was under scrutiny above.

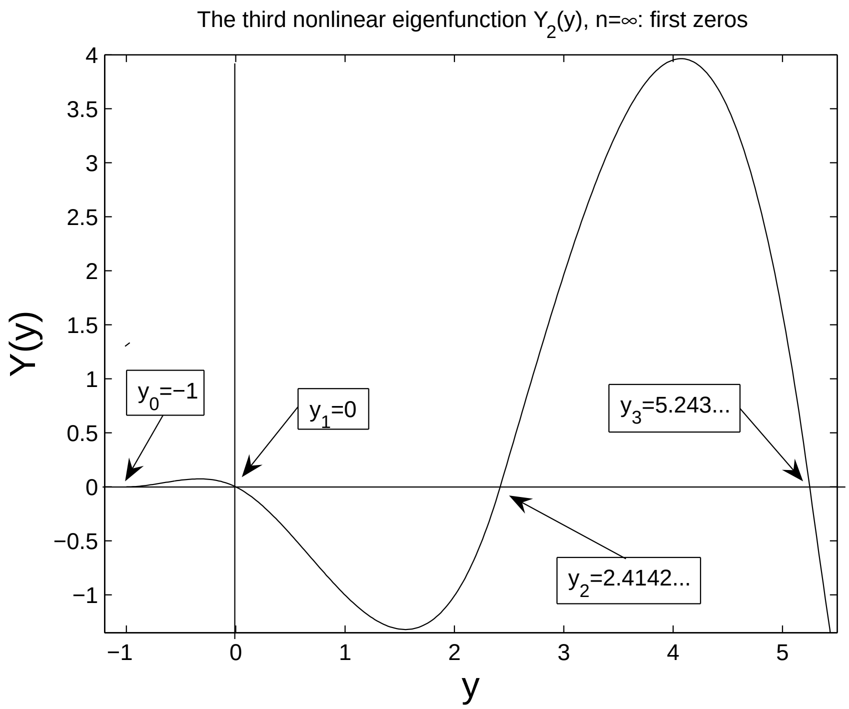

FIGURE 13. The third nonlinear eigenfunction $\tilde{Y}_2(y)$ as a solution of the problem (6.3), (4.22); $k = 1$, $l = 2$.

Recall that, using the reflections (2.8), simultaneously, we construct blow-up solutions

$$u_{\rm bl}(x,t) = (T-t)^{-\frac{1}{p-1}} f(y), \quad y = -\tfrac{x}{(T-t)^{\beta}}, \quad \text{where} \quad \beta = \tfrac{p-(n+1)}{(p-1)(2k+1)} \tag{7.4}$$

(since $\beta > 0$ for $p > n+1$, formally, this is a *single point blow-up*) of the *NDE with source*,

$$u_t = (-1)^{k+1} D_x^{2k+1}(|u|^n u) + |u|^{p-1}u, \tag{7.5}$$

so that proper profiles $f(y)$ describe both global and blow-up asymptotics of such NDEs.

In the case $n = 0$ in (7.3), we have a simpler semilinear ODE corresponding to the model (7.1). It is important also to note that, in addition to the homotopy path $n \to 0^+$ in (1.1), it is useful to apply an extra limit $p \to 1^+$. For the case $n = 0$ and $p = 1$ in (1.1), we reduce to the linear equation

$$u_t = (-1)^{k+1} D_x^{2k+1} u - u.$$

Using the substitution

$$u(x,t) = {\rm e}^{-t} v(x,t),$$

reduces it to the standard linear dispersion equation

$$v_t = (-1)^{k+1} D_x^{2k+1} v. \tag{7.6}$$

The extra scaling yields a semigroup (a group) with the infinitesimal generator $\mathbf{B}$ in (1.12):

$$v(x,t) = t^{-\frac{1}{2k+1}} w(y,t), \quad y = x/t^{\frac{1}{2k+1}}, \quad \tau = \ln t \quad \Longrightarrow \quad w_\tau = \mathbf{B} w. \tag{7.7}$$

Therefore, linear Hermitian theory developed in [4, § 4] (see also [6, §9]) can be used, and this leads to an efficient approach based on comparison of linear eigenfunction structures of (7.6) and the nonlinear ones of (1.1). Actually, this implies existence of a certain "homotopy" of those PDEs as $n \to 0^+$ (already noted in Section 3.6) and as $p \to 1^+$. Overall, this establishes a countable nature of the nonlinear eigenfunction family for (1.1), which is difficult to prove rigorously. We refer to [15], where such a homotopy is discussed for a related fourth-order PDE.

Thus, the continuous double limit, $n \to 0^+$ and $p \to 1^+$, by reducing to the LDE (7.6) with a *countable* set of rescaled eigenfunctions (2.13), in general, justifies that the ODE (7.3) admits a countable set of so-called p-bifurcation branches of solutions, which blow-up as $p \to (n+1)^+$. Further study of such nonlinear phenomena is of importance. For $n = 0$, this branching phenomenon has been studied in [4].

It is worth mentioning that, in general, such a *full* branching approach assumes existence of a countable number of 2D bifurcation surfaces defined in the 2D parameter space of (n, p), and the critical point $(0, 1)$ is expected to be a complicated singular "cusp" induced by such a collection of bifurcation surfaces. This is a very difficult bifurcation-branching phenomenon, which in 2D was not fully understood and remains an open problem. Therefore, below, we study another formal, but 1D, "nonlinear bifurcation" phenomenon in order to explain existence of a countable number of p-bifurcation branches in the problem (7.3).

7.2. **Local "nonlinear" bifurcations at critical exponents $p_l(n)$.** We consider the VSS solutions (7.2) of (1.1), and develop a formal *nonlinear* version of such a 1D p-bifurcation (branching) analysis for any fixed $n > 0$. As usual, according to classic branching theory [18, 23], a justification (if any) is performed for the equivalent semilinear integral equation with compact operators in suitable metrics. For simplicity, we present computations in the differential setting, which does not change anything essentially. Note that, for nonlinear odd-order operators, some issues of compactness can be rather tricky.

Let us compare the time-factor structure of the VSS (7.2) and that for the pure NDE in (2.2). It follows that the critical bifurcation exponents $\{p_l = p_l(n)\}$ are then determined from the equality of the exponents:

$$-\tfrac{1}{p_l-1} = -\alpha_l \quad \Longrightarrow \quad p_l(n) = 1 + \tfrac{1}{\alpha_l(n)} \quad l = 0, 1, 2, \ldots, \tag{7.8}$$

where $\alpha_l(n)$ are the critical exponents as in (2.27) obtained explicitly and further fully nonlinear ones that cannot be determined dimensionally (via conservation laws).

In particular, for the semilinear case $n = 0$, the eigenvalues $\alpha_l(0)$ are given by (2.12), and this leads to the critical bifurcation exponents

$$p_l(0) = 1 + \tfrac{2k+1}{l+1}, \quad l = 0, 1, 2, \ldots, \tag{7.9}$$

for the semilinear equation (7.1) studied in [4, § 7.4]. In the present nonlinear case with a fixed $n > 0$, such a standard linearised approach is not suitable.

However, first steps of this "nonlinear" bifurcation theory are straightforward. We next use an expansion relative to the small parameter $\varepsilon = p_l - p$, so that, as $\varepsilon \to 0$,

$$\alpha = \tfrac{1}{p-1} = \tfrac{1}{p_l-1-\varepsilon} = \alpha_l + \alpha_l^2 \varepsilon + O(\varepsilon^2)\,,$$
$$\beta = \tfrac{p-(n+1)}{(p-1)(2k+1)} = \tfrac{1-\alpha_l n}{2k+1} - \tfrac{n\alpha_l^2}{2k+1}\,\varepsilon + O(\varepsilon^2),$$
$$|f|^{p-1}f = |f|^{p_l-1-\varepsilon}f = |f|^{p_l-1}f(1-\varepsilon\ln|f| + O(\varepsilon^2)).$$

Note that, unlike the case (3.18), the last expansion has a clearer functional validity, since at $f=0$, there occurs standard issues of convergence, which makes sense suitable for passing to the limit in the integral operators.

Substituting these expansions into (7.3) and collecting $O(1)$ and $O(\varepsilon)$-terms yield

$$\mathbf{A}(f,\alpha_l) - |f|^{p_l-1}f + \varepsilon\mathcal{L}f + \varepsilon|f|^{p_l-1}f\ln|f| + O(\varepsilon^2) = 0,\ \mathcal{L} = -\tfrac{n\alpha_l^2}{2k+1}\,yD_y + \alpha_l^2 I, \tag{7.10}$$

where $\mathbf{A}(f,\alpha_l)$ is the nonlinear operator in (2.5). Recall that, at each nonlinear eigenvalue $\alpha = \alpha_l$, there exists the corresponding nonlinear eigenfunction f_l such that (2.7) holds. At least, we are going to use this conclusion, which was not completely proved. The fact is that the operator $\mathbf{A}(f,\alpha)$, with $\alpha = \alpha_l$ in (7.10) of the rescaled pure NDE, correctly describes the essence of a *"nonlinear bifurcation phenomenon"* to be revealed.

To this end, we use the additional invariant scaling of the operator $\mathbf{A}(f,\alpha)$ by introducing the new unknown function $F(\cdot)$ as follows:

$$f(y) = bF(y/b^{\frac{n}{2k+1}}), \tag{7.11}$$

where $b = b(\varepsilon) > 0$ is also an unknown parameter to be determined from a scalar branching equation and satisfying

$$b(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{7.12}$$

Substituting (7.11) into (7.10) and, under natural (but not proved) regularity assumptions on such expansions, omitting all higher-order terms (including the one with the logarithmic multiplier $\ln|b(\varepsilon)|$) yield

$$\mathbf{A}(F,\alpha_l) - b^{p_l-1}|F|^{p_l-1}F + \varepsilon\mathcal{L}F = 0. \tag{7.13}$$

Finally, we perform linearization about the nonlinear eigenfunction $f_l(y)$ by setting

$$F = f_l + Y.$$

This yields the following linear non-homogeneous problem:

$$\mathbf{A}'(f_l,\alpha_l)Y = b^{p_l-1}|f_l|^{p_l-1}f_l - \varepsilon\mathcal{L}f_l. \tag{7.14}$$

Here, the derivative is given by

$$\mathbf{A}'(f_l,\alpha_l)Y = (n+1)(-1)^{k+1}D_y^{2k+1}(|f_l|^nY) + \tfrac{1-\alpha_l n}{2k+1}\,Y'y + \alpha_l Y. \tag{7.15}$$

As usual in bifurcation theory, the rest of the analysis crucially depends on assumed good spectral properties of the linearised operator $\mathbf{A}'(f_l,\alpha_l)$, which are not easy at all, and, in fact, are much more complicated than that for the pair $\{\mathbf{B},\mathbf{B}^*\}$ in [4]. Many aspects of such a theory remain quite obscure. However, we proceed to explain the key

final results on possible bifurcations, and follow the same lines. A proper functional setting of this operator is more understandable in the present 1D case, where, using the behaviour of $f_l(y) \to 0$ as $y \to y_0^+$ and $y \to +\infty$, it is, at least formally, possible to check whether the resolvent is compact in a suitable weighted L^2 space. In general, this is a difficult open problem, especially since we are not aware of precise asymptotic properties of all the eigenfunctions f_l.

We assume that such a proper functional setting is available for $\mathbf{A}'(f_l, \alpha_l)$. Therefore, we deal with operators having solutions with "minimal" singularities at the boundary point of the support at $y = y_0 < 0$, where the operator is degenerate and singular. The same is assumed at $y = +\infty$, where the necessary admitted bundle of solutions should be identified to pose singular boundary conditions; see Naimark's monographs on ordinary differential operators [19, 20] as a guide.

Namely (cf. [4, § 4]), we assume that the linear odd-order operator $\mathbf{A}'(f_l, \alpha_l)$ has a discrete spectrum, and a complete and closed set of eigenfunctions denoted again by $\{\psi_\gamma\}$. We also assume that the kernel is finite-dimensional and we are able to determine the spectrum, eigenfunctions $\{\psi_\gamma^*\}$, and the kernel of the "adjoint" operator $(\mathbf{A}'(f_l, \alpha_l))^*$ defined in a natural way using the topology of the dual space L^2 (or, equivalently and possibly, of a space with an indefinite metric) and having the same point spectrum. The latter is true for compact operators in suitable spaces in more standard setting, [17, Ch. 4]. We also require that the bi-orthonormal eigenfunction subset $\{\psi_\gamma\}$ of the operator $\mathbf{A}'(f_l, \alpha_l)$ is complete and closed in a suitable weighted L^2-space (for $n = 0$, such results are available [4]). Note that, often, this "spectral collection" is too exhaustive in nonlinear operator theory; see Deimling [1, p. 412] for most general bifurcation results.

Thus, by a typical Fredholm-like alternative, the unique solvability of (7.14) requires the orthogonality of the inhomogeneous term therein to the $\ker \mathbf{A}'(f_l, \alpha_l)$. For simplicity, let it be 1D with the eigenfunction ϕ_l, so the right-hand side in (7.14) satisfies

$$b^{p_l-1}|f_l|^{p_l-1}f_l - \varepsilon \mathcal{L} f_l \perp \ker \mathbf{A}'(f_l, \alpha_l) = \mathrm{Span}\,\{\phi_l\}. \tag{7.16}$$

Then, multiplying (7.16) by ϕ_l^* in L^2 (or within the equivalent indefinite metric) yields the orthogonality condition (Lyapunov-Schmidt's algebraic branching equation [23, § 27]):

$$b^{p_l-1}\langle |f_l|^{p_l-1}f_l, \phi_l^*\rangle = \varepsilon \langle \mathcal{L} f_l, \phi_l^*\rangle. \tag{7.17}$$

Similar to (3.17), one needs to check whether the constants are non-zero:

$$\langle |f_l|^{p_l-1}f_l, \phi_l^*\rangle \ne 0 \quad \text{and} \quad \langle \mathcal{L} f_l, \phi_l^*\rangle \ne 0, \tag{7.18}$$

which is not an easy problem and can lead to some restrictions for such a behaviour, though is crucial for any hope to see a bifurcation point.

Under the conditions (7.18), the parameter $b(\varepsilon)$ in (7.11) for $p \approx p_l$ is given by

$$b(\varepsilon) \sim [\gamma_l(p_l - p)]^{\alpha_l(n)} \quad \big(\tfrac{1}{p_l-1} = \alpha_l\big), \quad \varepsilon = p_l - p, \quad \gamma_l = \tfrac{\langle \mathcal{L} f_l, \phi_l^*\rangle}{\langle |f_l|^{p_l-1}f_l, \phi_l^*\rangle}\,. \tag{7.19}$$

The direction of developing in p of each p_l-branch and whether the bifurcation is sub- or supercritical depend on the sign on the coefficient γ_l. This can be checked numerically

only, but, in general, we expect that $\gamma_l > 0$, so that these nonlinear bifurcations are *subcritical* and the p_l-branches exist for $p < p_l$.

Overall, the above formal analysis detects a number of key assumptions, which are necessary for such a nonlinear bifurcation to occur at the critical exponents p_l given by (7.8). Recall again that, for $n = 0$, a more rigorous justification of the corresponding linearised bifurcation analysis is done in [4], where a countable number of p-branches was shown to be originated at bifurcation points (7.9).

Overall, we claim that, for any $n \geq 0$, the ODE (7.3), with proper setting as $y \to \pm\infty$,

$$\boxed{\text{admits not more than } \textit{countable} \text{ number of } p\text{-branches of solutions,}} \tag{7.20}$$

which are originated at the critical exponents (7.8) (no rigorous proof is available still).

7.3. **Numerical experiments for $k = 1$.** We briefly attempt to find numerical solutions of the equation (7.3). As usual, we look at the lower-order case, $k = 1$. Once again, in order to remove the nonlinearity in the highest (third)-order operator, the substitution $Y = |f|^n f$ is used. This yields the semilinear third-order equation

$$f = |Y|^{-\frac{n}{n+1}} Y: \quad Y''' + \tfrac{p-(n+1)}{3(p-1)(n+1)} \, |Y|^{-\frac{n}{n+1}} Y' y + \tfrac{1}{p-1} \, |Y|^{-\frac{n}{n+1}} Y - |Y|^{\frac{p-(n+1)}{n+1}} Y = 0. \tag{7.21}$$

Due to the complexity of the equation, which remains to be of the third order, there is not much hope of solving this problem using a shooting method, since, in fact, we do not know in detail the "nonlinear bundle" as $y \to +\infty$. As usual and as in the semilinear case $n = 0$, there is a difficulty in finding the correct boundary points for $y > 0$, such that the correct oscillations are found.

Hence, we return to the BVP setting, trying to "optimise" and "minimise" the oscillatory bundle for $y \gg 1$. However, even using this approach, which was rather effectively implemented in the simpler semilinear case $n = 0$ in [4, § 6], it is difficult to produce reliable numerics, due to the highly nonlinear/oscillatory nature of the problem, even for smaller values of p and n. In addition, we must be careful when plotting, as we must avoid approaching any nonlinear bifurcation points in p given in (7.8), which actually are not known explicitly.

As a first example, in Figure 14, we present "almost converging" VSS profile for $p = 5$ and small $n = 0.1$. This example is of a particular importance: its tail for $y \sim 35$ is clearly not symmetric about $\{Y = 0\}$ and is positive. According to [4, § 6], such a solution *is not* a VSS profile in $\mathbb{R}$, and actually corresponds to some (obscure) BVP setting on a bounded interval, which is of no interest here. We should avoid such solutions in the future, even if these have been obtained with a perfect convergence up to the tolerances $\sim 10^{-3}$.

The next Figure 15 shows a better converging with a good tail VSS profile with the same $p = 5$ and a larger $n = 0.6$. We see that, for such larger n, the tail for $y \gg 1$ gets smaller and keeps being symmetric.

For larger $n \geq 1$, the numerics gets less reliable, though we have got a number of "almost converging" results. For instance, in Figure 16, this is done for $p = 10$ and $n = 1$. The tail is now larger for $y \sim 12$, and remains rather symmetric.

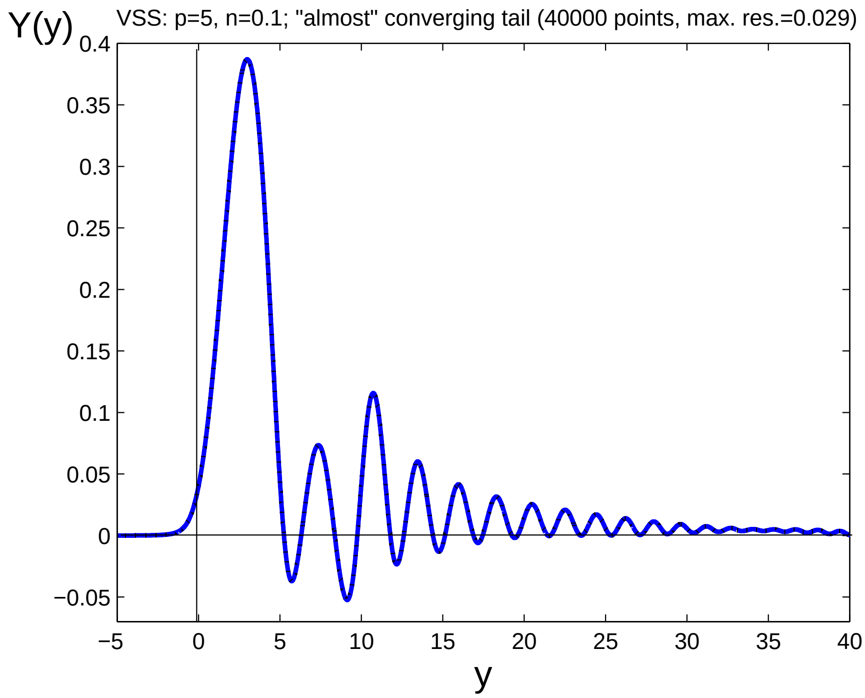

FIGURE 14. A VSS similarity solution $Y(y)$ of the ODE (7.21) for $p = 5$ and $n = 0.1$.

Further increasing n requires also increasing p. In the next Figures 17 and 18, we show VSS profiles for $p = 12$, $n = 2$ and $p = 16$, $n = 3$, respectively. In the former one, Figure 17, we present two different profiles (solid and dotted lines), showing that non-converging tails do not affect the convergence in the dominant positive part of the profiles.

Finally, we must admit that, since the correct "oscillatory bundle behaviour" as $y \to +\infty$ is still poorly understood, we cannot control it by choosing proper "minimal nonlinear components" (a proper symmetry is not enough; see [4, § 6.3]). Therefore, we must confess that, even having good enough numerical convergence, we cannot guarantee that the above numerical examples get into the countable family indicated in (7.20). In other words, without a full use of correct "minimal oscillatory bundle" as $y \to +\infty$ (meaning non-posing *any* condition at the singular end-point $y = +\infty$), the family of VSS profiles becomes continuous. Then, for any fixed value of $p > n + 1$, "solutions" $f(y)$ represent a continuous line, instead of a predicted at most countable subset of points, lying on the above p-branches.

On the other hand, it is plausible that those Figures correctly describe a general geometry of VSS profiles. In particular, we have always observed a strong "stability" of the first maximal positive hump, which turned out to be rather independent of the resulting tail for $y \gg 1$. In other words, we do note that, whilst the oscillatory part is difficult to obtain (and properly justify mathematically), the non-oscillatory structure is rather stable and does not change much, when changing the length in which the problem is

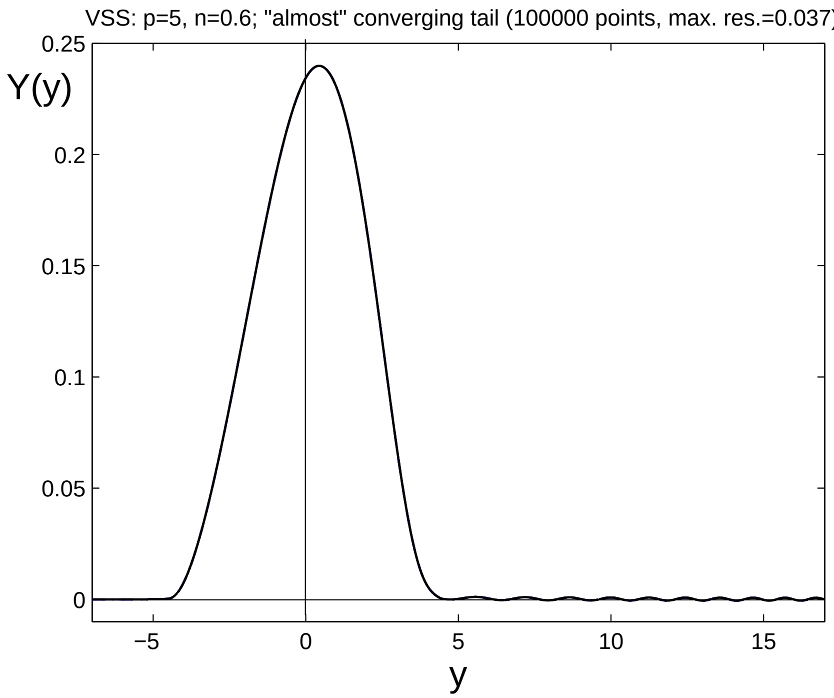

FIGURE 15. A VSS similarity solution $Y(y)$ of the ODE (7.21) for $p = 5$ and $n = 0.6$.

evaluated on. In particular, the numerical profile in the last Figure 18 was obtained with the worst *maximal residual* $= 10.23$ (actually meaning no convergence at all). However, we guarantee that this non-convergence takes place in the tail *only*, while the dominant positive part remains stable and "rather convergent", so we do not hesitate to present such a Figure here. However, in almost all other Figures, the convergence is much better and is not worse than $2-5\,\%$. It may be also noted that this confirms such oscillatory (at least, symmetric) tails, even when these are large in the amplitude, are essentially *zero* in some "weak" sense (which we do not want to specify at the moment).

7.4. **A nonlinear limit $n \to +\infty$: an example.** Finally, let us note that, as in Section 4, one way of finding proper oscillatory patterns $f(y)$, would be to look at the behaviour as $n \to +\infty$ and reduce the ODE to a simpler one. We present an example of such a limit along the following straight line on the $\{n, p\}$-plane:

$$p = n + 4 \to +\infty \quad \text{as} \quad n \to +\infty. \tag{7.22}$$

Then the ODE (7.21) reads

$$Y''' + \tfrac{1}{n+3}\,(|Y|^{-\frac{n}{n+1}} Y y)' - |Y|^{\frac{3}{n+1}} Y = 0, \tag{7.23}$$

so that, on scaling, one gets

$$Y = (n+3)^{-\frac{n+1}{n}} \tilde Y \quad \Longrightarrow \quad \tilde Y''' + (|\tilde Y|^{-\frac{n}{n+1}} \tilde Y y)' - (n+3)^{-\frac{3}{n}}\, |\tilde Y|^{\frac{3}{n+1}} \tilde Y = 0. \tag{7.24}$$

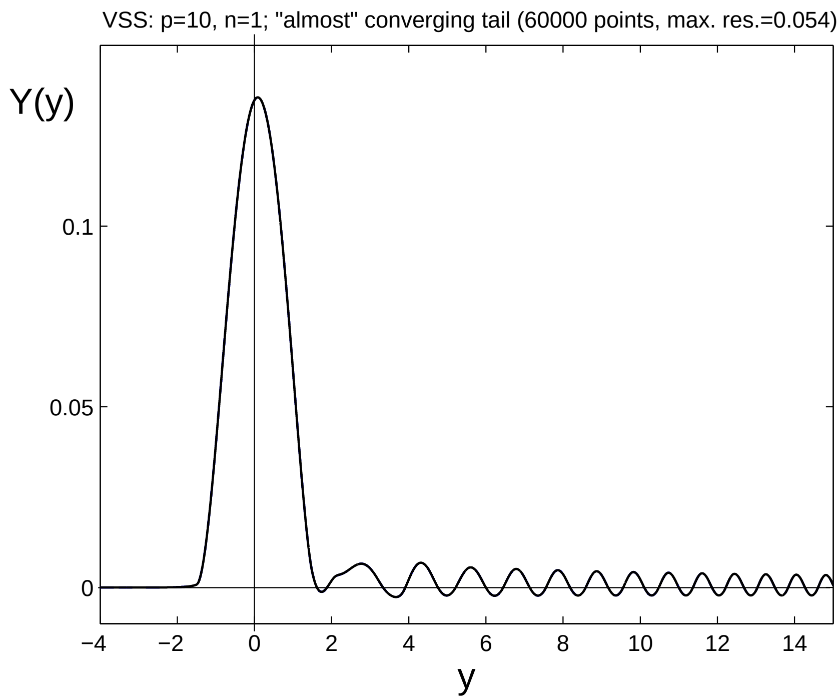

FIGURE 16. A VSS similarity solution $Y(y)$ of the ODE (7.21) for $p = 10$ and $n = 1$.

Passing to the limit $n \to +\infty$, in the ODE in (7.24), in the class of uniformly bounded solutions and using that $(n+3)^{-\frac{3}{n}} \to 1$, yield two terms as in the equation (4.4) with an extra linear one:

$$\tilde{Y}''' + (|\tilde{Y}|^{-\frac{n}{n+1}}\tilde{Y}y)' - \tilde{Y} = 0. \tag{7.25}$$

However, since unlike (4.4), this is a third-order ODE, an algebraic treatment of the first nonlinear eigenfunction $\tilde{Y}_0(y)$, as in Theorem 4.1, becomes rather illusive. Anyway, this shows a principal possibility to study the "nonlinear" limits $n \to +\infty$.

On the other hand, scaling also the independent variable y,

$$Y = C\,\tilde{Y}, \quad y = a\,\tilde{y}, \quad \text{and} \quad C = (n+3)^{-\frac{n+1}{n}} a^{\frac{3(n+1)}{n}}, \tag{7.26}$$

yields, instead of (7.24),

$$\tilde{Y}''' + (|\tilde{Y}|^{-\frac{n}{n+1}}\tilde{Y}\tilde{y})' - (n+3)^{-\frac{3}{n}}\, a^{3+\frac{9}{n}} |\tilde{Y}|^{\frac{3}{n+1}}\tilde{Y} = 0. \tag{7.27}$$

Therefore, passing to the limit $n \to +\infty$ in (7.27), after integrating once, we arrive precisely at the equation (4.4) provided that

$$a = a(n) \to 0^+ \quad \text{as} \quad n \to +\infty.$$

Therefore, we obtain the same nonlinear eigenfunctions for $l = 0, 1, 2$ via the above algebraic-geometric approach, but, according to (7.26), on expanding subsets in the independent $\tilde{y}$-variable.

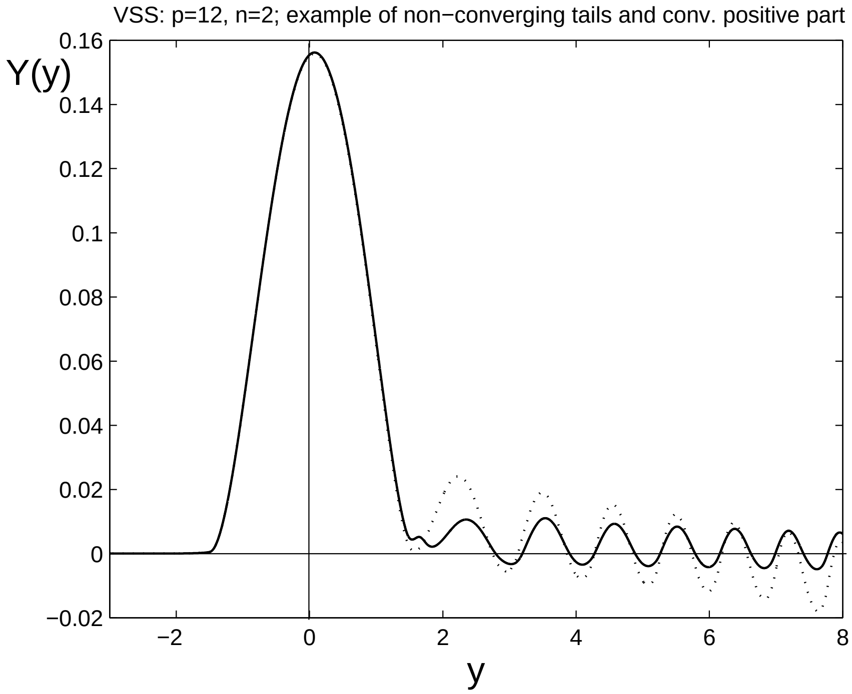

FIGURE 17. A VSS similarity solution $Y(y)$ of the ODE (7.21) for $p = 12$ and $n = 2$.

Recall that the above limit as $n \to +\infty$, with a possible study of branching as in Section 4.4, occurs along the straight line (7.22) (or in its "small neighbourhood") in the 2D parametric $\{n, p\}$-plane. Along other lines, the limits can be different and lead to other patterns $\tilde{Y}_1(y)$, $\tilde{Y}_2(y)$, etc.

Finally, overall, as we have seen, the ODE (7.21) represents a serious theoretical challenge with respect to both analytical study (n-branching, p-bifurcation diagrams, and p-branches; the latter are known for $n = 0$ [4, § 7], etc.), as well as even a numerical one.

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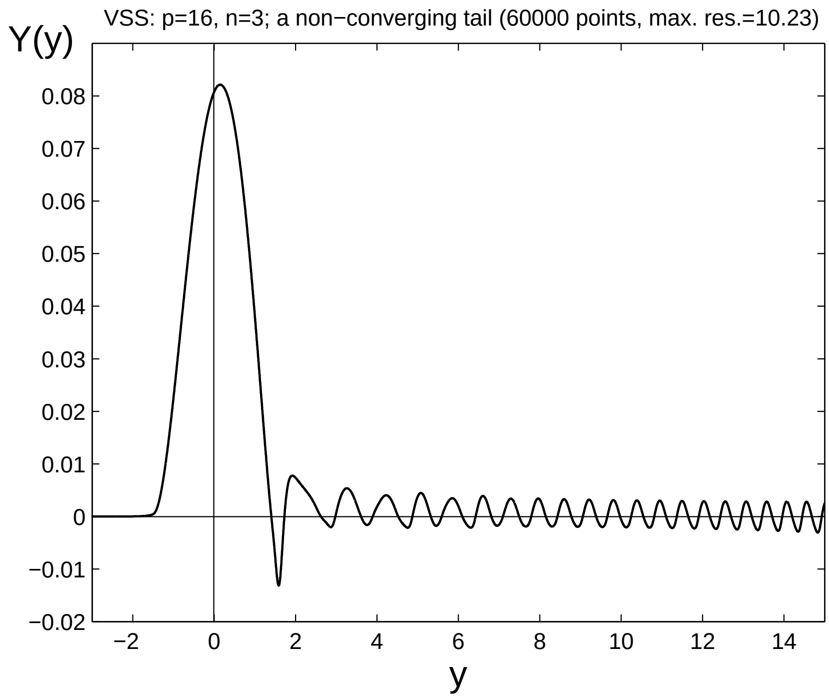

FIGURE 18. A VSS similarity solution $Y(y)$ of the ODE (7.21) for $p = 16$ and $n = 3$.

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